Special Problem 8 (a) (b) (please see your assignment sheet for the statement, to save paper!)

It will be enough to prove that each such f is absolutely continuous on $[0, \infty)$ (because we can get uniform continuity by choosing a collection consisting of just one interval). We are thus given $\epsilon > 0$.

By what was given, f is continuous at 0. Suppose that $x > 0$. To show that f is continuous at x, suppose that $x_o := x/2 \leq s \leq x \leq t$. Then (and we will use the inequality $(*)$ later, without reference to x):

$$f(t) - f(s) = t\frac{f(t)}{t} - s\frac{f(s)}{s} \leq t\frac{f(s)}{s} - s\frac{f(s)}{s} = \frac{f(s)}{s}(t-s) \leq \frac{f(x_o)}{x_o}(t-s),$$

so that

$$(*) \qquad f(t) - f(s) \leq \frac{f(x_o)}{x_o}(t-s) \;\text{ if }\; 0 < x_o \leq s < t.$$

Thus f is Lipschitz continuous on $[x_o, \infty)$, hence uniformly continuous there. Since x_o was an arbitrary positive number, this and our observation that f is continuous at 0 shows that f is continuous on $[0, \infty)$.

Let us now choose $x_o > 0$ such that $f(x_o) < \epsilon/2$. We also choose any $\delta > 0$ such that $\frac{f(x_o)}{x_o}\delta < \epsilon/2$.

Let $\{[a_n, b_n]\}$ be a countable non-overlapping collection of intervals contained in $[0, \infty)$, with $\displaystyle\sum_{n=1}^{\infty}(b_n - a_n) < \delta$.
We can divide this collection into three parts, some of which may be empty:
(1) The intervals with $a_n \geq x_o$;
(2) The intervals with $b_n \leq x_o$;
(3) An interval having x_o in its interior.

We can divide the interval $[a, b]$ in (3) (if there is one) in two: $[a, b] = [a, x_o] \cup [x_o, b]$. The two intervals do not overlap; their intersection has empty interior. Moreover, $f(b) - f(a) = f(b) - f(x_o) + f(x_o) - f(a)$, so the sum we are trying to estimate is not changed.

Thus we may assume that (3) does not happen, and so divide the sum we are estimating into two parts:

$$\sum_{n=1}^{\infty}|f(b_n) - f(a_n)| = \sum_{n=1}^{\infty}(f(b_n) - f(a_n)) = \sum_{a_n \geq x_o}(f(b_n) - f(a_n)) + \sum_{b_n \leq x_o}(f(b_n) - f(a_n)) =: S^+ + S^-.$$

By $(*)$ and our choice of δ,

$$S^+ \leq \sum_{a_n \geq x_o}\frac{f(x_o)}{x_o}(b_n - a_n) < \epsilon/2.$$

We next show that $\displaystyle\sum_{b_n \leq x_o}(f(b_n) - f(a_n)) \leq f(x_o) < \epsilon/2$, which will complete the solution. Since our intervals do not have to be ordered (there could be infinitely many of them between two points, and infinitely many on the other sides of each of the points in question), we proceed by taking a limit. In the sum

$$\sum_{\substack{b_n \leq x_o \\ n < N}}(f(b_n) - f(a_n))$$

the intervals $[a_n, b_n]$ *can* be put in left-to-right order. Suppose two of them are $[a, b]$ and $[c, d]$, with $b \leq c$. Then $f(d) - f(c) + f(b) - f(a) = f(d) - f(a) + f(b) - f(c) \leq f(d) - f(a)$ since $b \leq c$. It follows by induction that

$$\sum_{\substack{b_n \leq x_o \\ n < N}}(f(b_n) - f(a_n)) \leq f(\max b_n) - f(\min a_n) \leq f(x_o),$$

where the maximum and minimum are taken over the relevalnt numbers a_n and b_n. We may now let $N \to \infty$.