Special Problem 9: Due Dec 7

Chapter 3, # 18.
This Special Problem was intended as a redo of Special problem 8, with one change: "Replace the recursion formula of Exercise 16 by

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1} \tag{1}$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$."

The key to the solution presented here is the Difference of Powers Formula.

(a) By analogy with #16 we work with $p \geq 2$ ($p = 1$ is a trivial case) and select $x_1 > \alpha^{1/p}$. Some people considered the case $0 < x_1 < \alpha^{1/p}$ and most discovered that then $x_2 > \alpha^{1/p}$.

As we discovered when working on #16, we want to show that for all n, $x_n > \alpha^{1/p}$. We use induction starting at $n = 1$. Our statement is true for $n = 1$ by assumption. Next we assume that, for some $n \geq 1$, $x_n > \alpha^{1/p}$. We know that for every positive t, x^t increases strictly for $x > 0$. We will use this and the Difference of Powers Formula to relate "$x_n > \alpha^{1/p}$" and "$x_n^p > \alpha$," as follows:

$$x_n^p - \alpha = x_n^p - (\alpha^{1/p})^p = (x_n - \alpha^{1/p}) \sum_{k=0}^{p-1} x_n^{p-1-k} (\alpha^{1/p})^k = (x_n - \alpha^{1/p}) x_n^{p-1} \sum_{k=0}^{p-1} \left(\frac{\alpha^{1/p}}{x_n} \right)^k . \tag{2}$$

Then from (1) and a little rearranging we have

$$\begin{aligned} x_{n+1} - \alpha^{1/p} &= x_n - \alpha^{1/p} - \frac{1}{p} x_n \left(1 - \frac{\alpha}{x_n^p} \right) = x_n - \alpha^{1/p} - \frac{1}{p} x_n^{-p+1} (x_n^p - \alpha) \\ &= x_n - \alpha^{1/p} - \frac{1}{p} x_n^{-p+1} (x_n - \alpha^{1/p}) x_n^{p-1} \sum_{k=0}^{p-1} \left(\frac{\alpha^{1/p}}{x_n} \right)^k , \quad \text{by (2)}, \\ &= (x_n - \alpha^{1/p}) \left[1 - \frac{1}{p} \sum_{k=0}^{p-1} \left(\frac{\alpha^{1/p}}{x_n} \right)^k \right] > 0 \ \text{ since } \ x_n > \alpha^{1/p} \ \text{ and} \end{aligned} \tag{3}$$

since each of the p numbers $\left(\frac{\alpha^{1/p}}{x_n} \right)^k$ is less than one (except when $k = 0$, but we have $p - 1 > 0$). We notice, in passing that we have also shown that $x_n^p - \alpha > 0$ for all n.

Thus by induction, $x_n > \alpha^{1/p}$ for all n. In particular, $\{x_n\}$ is bounded below by $\alpha^{1/p} > 0$.

Next we show that $\{x_n\}$ decreases strictly. From (1) and some algebra

$$x_n - x_{n+1} = \frac{1}{p} \left(x_n - \frac{\alpha}{x_n^{p-1}} \right) = \frac{1}{p} \left(\frac{x_n^p - \alpha}{x_n^{p-1}} \right) > 0 \ \text{ for all } \ n.$$

Hence $\{x_n\}$ converges to $L := \inf_n x_n$. We apply Limit Theorems to (1) and find that $L = \frac{p-1}{p} L + \frac{\alpha}{p} L^{-p+1}$, or (factoring out L)

$$1 = \frac{p-1}{p} + \frac{\alpha}{p} L^{-p}, \ \text{ or } \ 1 = \alpha L^{-p} \Rightarrow L = \alpha^{1/p}.$$

Thus $x_n \to \alpha^{1/p}$.

(b) We next consider the errors, $\epsilon_n := x_n - \alpha^{1/p}$. From (3) we have

$$\begin{aligned} \epsilon_{n+1} &= x_{n+1} - \alpha^{1/p} = (x_n - \alpha^{1/p}) \left[1 - \frac{1}{p} \sum_{k=0}^{p-1} \left(\frac{\alpha^{1/p}}{x_n} \right)^k \right] = \epsilon_n \left[1 - \frac{1}{p} \sum_{k=0}^{p-1} \left(\frac{\alpha^{1/p}}{x_n} \right)^k \right] \\ &= \epsilon_n \frac{1}{p} \sum_{k=1}^{p-1} \left[1 - \left(\frac{\alpha^{1/p}}{x_n} \right)^k \right] \ \text{ because } \ 1 = \frac{1}{p} \sum_{k=0}^{p-1} 1 \ \text{ and } \ \left[1 - \left(\frac{\alpha^{1/p}}{x_n} \right)^0 \right] = 0. \end{aligned} \tag{4}$$

In (4) the term $1-\left(\frac{\alpha^{1/p}}{x_n}\right)^k$ can be rewritten, using Difference of Powers Formula:

$$1-\left(\frac{\alpha^{1/p}}{x_n}\right)^k = \frac{x_n^k-(\alpha^{1/p})^k}{x_n^k} = \frac{\epsilon_n}{x_n^k}\sum_{\ell=0}^{k-1} x_n^{k-1-\ell}(\alpha^{1/p})^\ell = \frac{\epsilon_n}{x_n}\sum_{\ell=0}^{k-1}\left(\frac{\alpha^{1/p}}{x_n}\right)^\ell < k\frac{\epsilon_n}{x_n} \tag{5}$$

unless $k=1$, where we have equality. Combining (4) and (5) gives (since $p-1>0$)

$$\epsilon_{n+1} = \epsilon_n\frac{1}{p}\sum_{k=1}^{p-1}\left[1-\left(\frac{\alpha^{1/p}}{x_n}\right)^k\right] < \epsilon_n\frac{1}{p}\sum_{k=1}^{p-1}k\frac{\epsilon_n}{x_n} = \frac{\epsilon_n^2}{x_n}\frac{p-1}{2} = \frac{(p-1)}{2x_n}\epsilon_n^2 < \frac{(p-1)}{2\alpha^{1/p}}\epsilon_n^2. \tag{6}$$

Next we put $\beta := 2\alpha^{1/p}/(p-1)$ and this gives us the analog of what we had in #16: $\epsilon_{n+1} < \beta\left(\frac{\epsilon_n}{\beta}\right)^2$.

We can now use the argument used in #16 to conclude that,

$$\text{with } \beta = \frac{2\alpha^{1/p}}{p-1}, \quad \epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}, \quad n \geq 1. \tag{7}$$

(c) An example. The larger p is, the better our initial guess has to be, because β now has the divisor $p-1$. Take $\alpha=3$ and $p=3$. To make the ideas in #16 work, we want to "rationalize" in (2), solved for $(x_1-3^{1/3})$ in terms of x_1^3-3. This rationalization uses the Difference of Powers Formula and the fact that $3^{1/3}>1$:

$$\epsilon_1 = x_1-3^{1/3} = \frac{x_1^3-(3^{1/3})^3}{x_1^{3-1}\sum_{k=0}^{3-1}\left(\frac{3^{1/3}}{x_1}\right)^k} = \frac{x_1^3-3}{\sum_{k=0}^{3-1}x_1^{3-1-k}3^{k/3}} < \frac{x_1^3-3}{\sum_{k=0}^{3-1}x_1^{3-1-k}} = \frac{x_1^3-3}{x_1^3-1}(x_1-1).$$

From (7) we see that $\beta = 3^{1/3} > 1$. If we try $x_1=2$ we get $\epsilon_1/2x_1 < \frac{x_1^3-3}{2x_1(x_1^3-1)} = \frac{5}{28}$. Though this is less than 1/4, it is not less than 1/10. We need $x_1<2$, but not too small. I tried $x_1=4/3$ and $x_1=5/3$. Of these values, the first is too small, the second too large. Halfway between the last two tries is $x_1=3/2$. This gives

$$\epsilon_1/\beta < \frac{x_1^3-3}{\beta(x_1^3-1)}(x_1-1) < \frac{x_1^3-3}{x_1^3-1}(x_1-1) = \frac{\frac{27}{8}-\frac{24}{8}}{\frac{19}{8}}\frac{1}{2} = \frac{3}{38} < \frac{1}{10}.$$

We then have the "same" estimates as we had in #16, but we can use 2 now instead of 4; the powers of 10 will be the same.