Math 4606, Summer 2005 **Special Problems 3 and 2:** solutions

Special Problem 3: Due July 8

See §**2.8** *#10: Answer the "why?" question in the first ¶, then use the first ¶ to show that for all* $A > 0$ *there is a unique positive solution* c *of the equation* $c^2 = A$. *For double credit, show that for all* $A > 0$ *and for all positive integers* n *that there is a unique positive solution* c *of the equation* $c^n = A$. *Induction won't work!*

This solution is only for the $n = 2$ part. See me to talk about the bonus part. Moreover, this solution has too much detail! I nevertheless put it in because I suspect some of you may need the excess detail. I hope it helps!

We first define the set $S := \{x > 0 : x^2 < A\}$. The plan is to show that $c := \sup S$ exists in $\mathbb{R}$ and that $c^2 = A$. To show that $\sup S$ exists we hope we can show that S is not empty and that S is bounded above because then we can use the Completeness Axiom. To make a choice of $0 < x \in S$ we have to be clever, because if $A < 1$, then $1 \notin S$. We need an x that, when squared stay less than A. But if $0 < x < 1$, then $0 < x^2 = x \cdot x < 1 \cdot x = x$, multiplying thru the inequalities $0 < x < 1$ by the positive number x. Hence, if $0 < A < 1$ we can take $x = A$ and it will be true that $x^2 = A^2 < A$. It would also work if we used $x = \dfrac{A}{1+A}$, because now we have $x < 1$ and $x < A$. Indeed, *this* x works even if $A \geq 1$. The reason is that $x^2 < \dfrac{A}{1+A}$ because $x = \dfrac{A}{1+A} < 1$, and because $\dfrac{A}{1+A} < A$, because the denominator is greater than one. Hence if we choose $x := \dfrac{A}{1+A}$ then $0 < x$ and $x^2 < A$. Thus S is not empty. Since we have found a *positive* element of S it is also true that every upper bound of S is positive.

To show that S is bounded above, we find it handy to notice this *Observation: if* $0 < x$ *and* $0 < y$ *then* $x < y$ *if and only if* $x^2 < y^2$. This needs a two-way proof. Thus, if $0 < x < y$ we multiply thru by x: $0 < x^2 < x \cdot y$. But $x \cdot y < \frac{y}{x} \cdot x \cdot y = y^2$ because $\frac{y}{x} > 1$. Thus $0 < x < y \Rightarrow x^2 < y^2$. On the other hand, if $0 < x$ and $0 < y$ and $x^2 < y^2$ then $0 < y^2 - x^2 = (y - x)(y + x)$. Since $x + y > 0$ we can multiply thru by $1/(y + x)$ to get $0 < y - x$, or $x < y$. (**Note:** can you prove axiomatically that $x > 0$ if and only if $x^{-1} > 0$?) We knew that $0 < x$, so we finally have $0 < x < y$. You could have used the *Observation* without proof.

Let's show that if $0 < x$ and $x^2 < A$, then $x < A + (1/2)$. Well, we know that $(A + (1/2))^2 = A^2 + A + 1/4 > A^2 + x^2 + 1/4 > x^2$, so by the *Observation* in the last ¶ we see that $x < A + (1/2)$, so S is indeed bounded above. Thus $c := \sup S$ exists in $\mathbb{R}$. Moreover, because we know a positive element of S, $c > 0$.

We have to show that $c^2 = A$. We show that assuming either of $c^2 < A$ and $c^2 > A$ gives a contradiction.

If we *assume* $c^2 < A$ then because c is positive, $c \in S$. The proof for $A = 3$ gives us an idea: Find an $h > 0$ so small that $(c + h)^2 < A$ is true. The $c + h$ would be in S and at the same time larger than c, which is an upper bound for S. Well, we know that

$$\lim_{h\to 0}(c+h)^2 = \lim_{h\to 0} c^2 + \lim_{h\to 0} 2ch + \lim_{h\to 0} h \cdot \lim_{h\to 0} = c^2 + 2c \cdot 0 + 0 \cdot 0 = c^2,$$

so, taking $\epsilon := A - c^2 > 0$, there exists $\delta > 0$ such that $0 < h < \delta \Rightarrow 0 < (c + h)^2 - c^2 < \epsilon = A - c^2$, so $(c + \delta/2)^2 < A$. We let $h = \delta/2$. This gives one of the contradictions we wanted. **Note:** the first inequality after the $\Rightarrow$ came from the *Observation.*

Next, if we *assume* $c^2 > A$ then taking $\epsilon := c^2 - A > 0$, there exists $\delta > 0$ such that $\delta < c$ and such that $|h| < \delta \Rightarrow |(c + h)^2 - c^2| < \epsilon = c^2 - A$. We now let $h = -\delta/2$. Then $|(c + h)^2 - c^2| = c^2 - (c + h)^2 > 0$ (again, by the *Observation,* because now $c + h = c - \delta/2 > 0$). Therefore $c^2 - (c - \delta/2)^2 < \epsilon = c^2 - A$. Hence $(c - \delta/2)^2 > A$. But $c - \delta/2 < c$. Thus there exists $x \in S$ such that $c - \delta/2 < x$. But trhen by the *Observation,* and by what we have done, and by the Definition of S,

$A < (c - \delta/2)^2 < x^2 < A$. This is a contradiction. Thus (by Trichotomy) $c^2 = A$.

Special Problem 2: Due June 24

Solve exactly one of the following problems (1), (2); i.e., turn in exactly one solution:
(1) Prove that for all $x_1 \in \mathbb{N}$ *and for all* $x_2 \in \mathbb{N}$, $x_1 + x_2 \in \mathbb{N}$ *and* $x_1 \cdot x_2 \in \mathbb{N}$.

(2) Prove that for all $x_1 \in \mathbb{N}$ *and for all* $x_2 \in \mathbb{N}$, $|x_1 - x_2| \in \mathbb{N}$.

(1): Let x_2 be an arbitrary natural number. We define

$$G_{x_2} := \{x \in \mathbb{N} : x + x_2 \in \mathbb{N}\}.$$

Then $0 \in G_{x_2}$ because $x_2 \in \mathbb{N}$ and $0 + x_2 = x_2 \in \mathbb{N}$. Next, if $x \in G_{x_2}$ then

$$(x+1) + x_2 = x + (1 + x_2) = x + (x_2 + 1) = (x + x_2) + 1 \in \mathbb{N}$$

because $x + x_2 \in \mathbb{N}$ and $\mathbb{N}$ is inductive. Then $x + 1 \in G_{x_2}$. Thus G_{x_2} is inductive. Since G_{x_2} is an inductive set contained in $\mathbb{N}$, $G_{x_2} = \mathbb{N}$. Since x_2 was arbitrary, the first part of (1) is solved. We will use this part in the proof of the other part!

In the "other part," about products, we let x_2 be an arbitrary natural number and this time define

$$G_{x_2} := \{x \in \mathbb{N} : x \cdot x_2 \in \mathbb{N}\}.$$

We have shown that $0 \cdot x = 0$ for all x in a field, and $\mathbb{R}$ is a field, so $0 \cdot x_2 = 0 \in \mathbb{N}$. Thus $0 \in G_{x_2}$.

Next, if $x \in G_{x_2}$, so that $x \cdot x_2 \in \mathbb{N}$, then

$$(x+1) \cdot x_2 = x \cdot x_2 + 1 \cdot x_2 = (x \cdot x_2) + x_2 \in \mathbb{N}$$

by the first part of this problem. This shows that G_{x_2} is an inductive set contained in $\mathbb{N}$, so $G_{x_2} = \mathbb{N}$.

Since x_2 was arbitrary, the second part is solved. This completes the solution of (1).

(2): This part can be proved by looking for inductive sets. But we prove "inductive-ness" using inductions. Again we let x_2 be an arbitrary natural number and define

$$G_{x_2} := \{x \in \mathbb{N} : |x - x_2| \in \mathbb{N}\}, \quad \text{noticing right away that } G_0 = \mathbb{N} \text{ and that } 0 \in G_{x_2} \text{ for all } x_2 \in \mathbb{N}.$$

In what follows we thus assume that $x_2 > 0$. Then $x_2 \in G_{x_2}$ because $0 \in \mathbb{N}$. If $x \geq x_2$ and $x \in G_{x_2}$ then

$$|(x+1) - x_2| = x - x_2 + 1 = |x - x_2| + 1 \in \mathbb{N}.$$

Thus G_{x_2} contains all natural numbers $x \geq x_2$.

Lemma: *If* $y \in \mathbb{N}$ *and* $y > 0$ *then* $y - 1 \in \mathbb{N}$. This is certainly true when $y = 1$, the smallest positive member of $\mathbb{N}$, by a result previously proved. If true for some $y \in \mathbb{N}^+$, then $y + 1 \in \mathbb{N}$ and $(y+1) - 1 = y \in \mathbb{N}$. The Lemma is proved!

Now suppose that $x \in \mathbb{N}$ and $x < x_2$. Then $0 < |x - x_2| = x_2 - x \in \mathbb{N}$, and

$$|(x+1) - x_2| = |(x - x_2) + 1| = (x_2 - x) - 1 \in \mathbb{N}, \quad \text{by the Lemma.}$$

Thus $x + 1 \in G_{x_2}$ whenever $x \in G_{x_2}$. We already noticed that $0 \in G_{x_2}$, so $G_{x_2} = \mathbb{N}$ because G_{x_2} is an inductive set contained in $\mathbb{N}$. As x_2 was arbitrary, (2) is solved.