**Math 5615H, Fall 2003** **(3.7):** The set of subsequential limits is closed.

**Proof**
Let $E^*$ denote the set of points of $X$ that are limits of a subsequence of $\{p_n\}$. Suppose that $x \notin E^*$. We have to show that there exists $\delta > 0$ such that $B_\delta(x) \subseteq (E^*)^c$. Let us proceed by contradiction. This means we are assuming that for every $\delta > 0$, $B_\delta(x) \cap E^* \neq \emptyset$. If $q \in E^* \cap B_\delta(x)$ there exists $\eta > 0$ such that $B_\eta(q) \subseteq B_\delta(x)$. Hence there exists $p_n$ such that $d(p_n, q) < \eta$ so some $p_n \in B_\delta(x)$. We have shown that every neighborhood of $x$ contains a term of the sequence $\{p_n\}$.

Intuitively, that suggests there ought to be a subsequence that tends to $x$, thus giving us the contradiction we seek.

Let's construct such a subsequence, using
**The Recursion Theorem:** *Let $Y$ be a non-empty set, and suppose that $y_1 \in Y$. Suppose also that $H : Y \to Y$ is a function. Then there exists a unique sequence $R : \mathbb{Z}^+ \to Y$ such that $R_1 = y_1$ and such that for all $n \in \mathbb{Z}^+$, $R_{n+1} = H(R_n)$.*

The sequence we get from the Recursion Theorem won't just be a subsequence of $\{p_n\}$. We have to set it up so that the set $Y$ and the function $H$ describe what we do at each "stage of the construction."

We define a set $S_\delta := \{n \in \mathbb{N} : p_n \in B_\delta(x)\}$. We have shown that (under our contradiction assumption) $S_\delta \neq \emptyset$ for all $\delta > 0$. Therefore $S_\delta$ is a non-empty subset of $\mathbb{N}$. Let's *try* defining the set $Y$ in the Recursion Theorem to be

$$Y := \{(\delta, S) : \delta > 0,\ S \subseteq \mathbb{N},\ S \neq \emptyset, \text{ and } S = S_\delta\}, \quad \text{so that } Y \text{ is a nonempty set of subsets of } \mathbb{N} \times \mathbb{R}^+.$$

Our function $H$ will start with $(r, S_r)$, which we get using $r > 0$ and our way of defining $S_r$. We find the minimum member of $S_r$ and call it $n(r)$. The signifigance of this: $p_{n(r)} \in B_r(x)$, and for *every* $p_n \in B_r(x)$, $n > n(r)$. We set $r' := d(x, p_{n(r)})/2$. We finally put $H(r, S) := (r', S_{r'})$. We already know that $S_{r'} \neq \emptyset$, so $(S_{r'}, r') \in Y$.

We start with $(S_1, 1) \in Y$ and apply the Recursion Theorem. We thus get a sequence of sets $E_n := S_{r_n}$ and we can define (without using the Recursion Theorem) $n_i = \min E_i$. By the way we defined our function $H$, $E_{i+1} \subseteq E_i$ and $n_i \notin E_{i+1}$. Hence $n_{i+1} > n_i$ and $p_{n_i} \in B_{r_i}(x)$. Finally, we note that (by an induction left for you to check) $r_i \leq 2 \cdot 2^{-i}$, so $p_{n_i} \to x$. But then $x \in E^*$, a contradiction. You may have to use: $n_i \geq i$.

The reason that the "usual" way of constructing a sequence by induction is wrong is the phrase "suppose $n_i$ have been selected, for $1 \leq i \leq n$." There is no specification that the construction has been done in a particular way, whereas the Recursion Theorem does exactly that.

*Naive Set Theory,* by Paul Halmos, discusses the Recursion Theorem on page 48. Halmos uses $\omega$ for $\mathbb{N}$ and uses $n^+$ (the *successor* of $n$) in place of $n+1$. The proof there is short! Another note on the Recursion Theorem should be on the Web now too, that gives a proof of the Recursion Theorem (it expands Halmos's proof) and includes, as an example, "cleaning up" Rudin's proof of **3.7**.