

The questions listed here are from an old final for Math 5612, approximately the same as the first 2/3 of Math 5615 I put them up now so you can get started reviewing...

1(9) Define metric space. Prove that, in a metric space  $X$ , for all  $x, y$  and  $z$  in  $X$ ,  $|d(x, y) - d(x, z)| \leq d(y, z)$ .

2(10) Define convergent sequence for metric spaces. Suppose that  $\{x_n\}$  and  $\{y_n\}$  converge to the same limit,  $p$ , in a metric space  $X$ . Prove that, for any sequence  $\{n_k\}$  of positive integers, the merged sequence  $\{w_j\}$  formed by taking the first  $n_1$  terms from  $\{x_n\}$ , then the first  $n_2$  terms from  $\{y_n\}$ , then the next  $n_3$  terms from  $\{x_n\}$ , and so on, alternating between  $\{x_n\}$  and  $\{y_n\}$ , converges to  $p$  also.

Recall that, in a metric space, a point  $p$  is a boundary point of a set  $E$  if every neighborhood of  $p$  contains a point in  $E$  and a point not in  $E$ .

3(8) Write out the definition of "boundary point" in logic notation, or in the same way, in words. Supply all the quantifiers. Also, give a definition of "neighborhood" in the same manner.

4(12) Define interior point of a set  $E$  in a metric space  $X$ . Prove that, given a subset  $E$  of  $X$ , every point  $p$  of  $X$  is either a boundary point of  $E$ , an interior point of  $E$ , or an interior point of the complement of  $E$ .

5(9) Define Cauchy sequence in a metric space. Prove that a sequence of real numbers

6(10) Prove: for every pair of real numbers  $a$  and  $b$ , such that  $a < b$ , there exists an irrational number  $c$  such that  $a < c < b$ .

7(15) State Cauchy's Condensation Test. Prove that, if  $p > 1$ , then there exist positive constants  $A_p$  and  $B_p$  such that, for all  $m$  in  $\mathbb{Z}^+$ ,  $A_p m^{1-p} \leq \sum_{n=m}^{\infty} \frac{1}{n^p} \leq B_p m^{1-p}$ . You may use, without proof, that for all  $m$  in  $\mathbb{Z}^+$ , there exists a unique  $m$  in  $\mathbb{Z}^+$  such that  $2m - 1 \leq m < 2m$ .

8(8) Define compact set (in a metric space  $X$ ). What property of its infinite subsets is equivalent to  $X$  being compact?

9(6) Suppose that  $A(BorC)$  is false. What can be said about the truth-values of  $A, B$  and  $C$ ? Answer the same question if  $A(BorC)$  is true.

10(10) Let  $b$  denote the set of all bounded sequences of complex numbers. For sequences  $x = \{x_n\}$  and  $y = \{y_n\}$  in  $b$ , define  $d(x, y) := \sup_{n \in \mathbb{N}} |x_n - y_n|$ . Prove that  $b$  is a metric space, with this  $d$ .

11(10). With  $b$  and  $d$  as in # 10, prove that  $b$  is a complete metric space.

12(10). With  $b$  and  $d$  as in # 10, let  $\mathbf{0}$  denote the sequence  $0, 0, \dots$ . Prove that  $\{x \in b : d(x, \mathbf{0}) \leq 1\}$  is not a compact set.

13(10). For which complex numbers  $z$  does the series  $\sum_{n=0}^{\infty} c_n z^n$  converge, with  $c_n = 1/2^n$  for infinitely many  $n$ , and  $c_n = 1/3^n$  for infinitely many  $n$ , and  $c_n = 0$  for infinitely many  $n$ , but with  $c_n$  taking no other values. Why?

14(8) Invent a truth-table for "A unless B," and express "A unless B" in terms of "and," "or," and "not." Explain your answer!

15(15) Prove directly that a closed, bounded interval  $[a, b]$  is a compact set in  $\mathbb{R}$ , with its usual topology.

6(6) State the Archimedean Property for the Real Number System. State an analogous version for powers.

17(8) Suppose that  $S$  is a bounded open set of real numbers. Suppose  $0 \in S$ . Show that  $\inf S$  and  $\sup S$  both exist, and that neither  $\inf S$  nor  $\sup S$  belongs to  $S$ .

18(15) Prove that the set of real numbers is uncountable.

19(12) Define open set and closed set of a set  $E$  in a metric space  $X$ . Then state and prove different, but equivalent conditions for sets to be open, or closed.

20(8) Define limit point, of a set  $E$  in a metric space  $X$ . Write out the *denial* of the statement " $p$  is a limit point of  $E$ " in logic notation, or in the same way, in words. Supply all the quantifiers.