Indicate your approach! Show your work! Choose Questions carefully. Good Luck! If you want to use a Theorem, give its name, identify it by "what it says," and be sure you check out the hypotheses! You may use the result of one Question in *another* **Question.** *There are* 249 *points.*

(1) [8] Let X be a metric space, with metric d, and let x_o be a fixed point in X. Prove that $f(x) := d(x, x_o)$ is uniformly continuous on X.

(2) [6] Define *connected set.* Define any "new" terms that you use in your definition.

(3) [6] Define "continuous at a point," and "continuous on a set," in the context of metric spaces.

(4) [10] State the Intermediate Value Theorems for continuous functions and for derivatives. Show that, if $g'(x)$ exists in an interval, and is never zero in that interval, then g is monotone in the interval.

(5) [15] State and prove the (usual) Mean Value Theorem.

(6) [10] Let $f : X \to Y$, where X and Y are metric spaces. Prove, using epsilons and deltas, that if f is continuous at a point x_o, then for every sequence $\{x_n\}$ such that (as $n \to \infty$,) $x_n \to x_o$, $f(x_n) \to f(x_o)$.

(7) [8] Suppose that $f : [0, 1] \to [0, 1]$ is continuous. Prove that f has a fixed point.

(8) [10] State, and prove the theorem that gives the relation between continuity of a function f on a metric space X and open subsets of the range (metric) space Y.

(9) [10] Define *differentiable at* x_o. State and prove tha Chain Rule.

(10) [18] Find $\sum_{n=4}^{\infty} \frac{2^n}{3^n}$, $\sum_{n=1}^{\infty} \frac{n+3}{n(n+1)(n+2)}$, and the radius of convergence of $\sum_{n=4}^{\infty} \frac{z^{n^2}}{3^n}$.

(11) [10] Define "$\limsup$," for sequences, and show that $\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n$.

(12) [15] Suppose that $f(x)$ is continuous and real-valued on $[0, 1]$, and differentiable on $(0, 1)$. Suppose also that $f(x)^2 + f'(x)^2 > 0$ for all x in $(0, 1)$. Prove that the zeroes of f (solutions of the equation $f(x) = 0$) are isolated in $(0, 1)$. Hint: What do we know about $f(x)$ near where $f'(x_o) \neq 0$?

(13) [10] State l'Hospital's Rule, and explain how the Cauchy Mean Value Theorem is used to prove it in the case when both functions tend to zero.

(14) [20] Suppose that $f(x)$ is a differentiable real-valued function defined for $x > 0$. Suppose further that $L := \lim_{x \downarrow 0} f'(x)$ exists and is finite. Show that f can be extended to a function on $\mathbb{R}$ that is continuous and differentiable on $\mathbb{R}$.

(15) [15] State the Cauchy Mean Value Theorem and prove it.

(16) [10] Suppose that $\{a_n\}$ is a real sequence and that $\sum a_n$ converges, but $\sum |a_n|$ does not. Prove that neither of $\sum \max(a_n, 0)$ and $\sum \min(a_n, 0)$ can converge either.

(17) [20] Suppose that, for each n, $\sum_{k=1}^{\infty} a_{nk}$ is a series with non-negative terms. Suppose also that for each k, a_{nk} increases to a_k as n increases to ∞. Prove that $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty} a_k$.

(18) [8] Suppose that f and g are continuous on $[a, b]$. Prove that $cf(x) + dg(x)$ is continuous on $[a, b]$, where c and d are numerical constants.

(19) [10] State (carefully) and prove the Theorem that the composition of continuous functions is continuous.

(20) [10] Is the function given by $f(x, y) := x^2y^3/(x^2 + y^4)$ if $(x, y) \neq (0, 0)$ and $f(0, 0) := 0$ continuous at $(0, 0)$? Why?

(21) [10] Prove that a set E in a metric space X is closed if and only if E contains all its boundary points.