Indicate your approach! Show your work! Good Luck! There are 4 pages, and 100 points.

(1) [10] State and prove Cauchy's Condensation Test.

(2) [12] Let $X := [0,+\infty) \cup \{+\infty\} =: [0,+\infty]$. Define the function $d : X \times X \to [0,+\infty)$ by the formulas $d(x,y) := \left|\frac{x}{1+x} - \frac{y}{1+y}\right|$, if both of x and y belong to $[0,+\infty)$;
$d(x,y) := \frac{1}{1+x}$, if $x \in [0,+\infty)$ and $y = +\infty$; $d(x,y) := \frac{1}{1+y}$, if $y \in [0,+\infty)$ and $x = +\infty$;
$d(x,y) := 0$, if $x = +\infty = y$. Assume that you have already proved that $d(x,y)$ is a metric on $[0,+\infty)$.
Complete the proof that $d(x,y)$ is a metric on $[0,+\infty]$.
You may use these statements without proof: $\frac{x}{1+x}$ increases with x on $[0,+\infty)$; $\frac{x+y}{1+x+y} \leq \frac{x}{1+x} + \frac{y}{1+y}$ for all real $x \geq 0$ and for all real $y \geq 0$; $1 + |x-y| \leq 1 + x + y + xy$ for all real $x \geq 0$ and for all real $y \geq 0$.

(3) [10] Let X be a metric space, with metric d, in which every infinite set has a limit point. Give a self-contained proof that there exists a real number M such that, for all pairs of points x and y in X, $d(x, y) \leq M$.

(4) [8] Define "compact set." Explain your terms!

(5) [10] This is a continuation of Question 2. Prove that X is a compact metric space. You may use, without proof, the result that every closed and bounded interval $[a, b]$ in $[0, +\infty)$ is compact in the metric space X.

(6) [12] Let $z \in \mathbb{C}$. Prove that $\{z^n\}$ converges to a finite limit if and only if $|z| < 1$ or $z = 1$.

(7) [10] Suppose p is a limit point of a set E in a metric space X, but p is not a boundary point of E. Show that p is an interior point of E.

(8) [9] Construct an example of a sequence of complex numbers that is bounded, has infinite range, and whose set of subsequential limits has three points.

(9) [8] Define *monotone sequence* fully, and show that a monotone sequence converges in $\mathbb{R}$ if and only if it is bounded.

(10) [11] Discuss the convergence behavior of the geometric series. *This will be scored competitively!*