Here are some Definitions and Theorems about sequences, and proofs of the Theorems. There will be "(why?!)'s" here and there that are Exercises you can probably do without writing anything down. There will be Exercises too, that you'll probably have to do some writing to carry out. These are also here as proof examples.

(01) **Theorem:** *If* $\{x_n\}$ *is a sequence of points in a closed set* S *and* $\lim_{n\to\infty} x_n$ *exists then* $\lim_{n\to\infty} x_n \in S$.

Proof: Let $L := \lim_{n\to\infty} x_n$. Suppose that $L \notin S$. Then because S^c is open there exists $h > 0$ such that the open interval $(L-h,\, L+h)$ is contained in S^c. This gives a contradiction. Details: since $\lim_{n\to\infty} x_n = L$ we use $h > 0$ as the $\epsilon > 0$ in the "in logic" version of the statement $\lim_{n\to\infty} x_n = L$. Then there exists $K \in \mathbb{N}$ such that $n \ge K$ implies that $|x_n - L| < h$. But then "$x_n \in S$ and $x_n \in S^c$" is true. This is a contradiction. Thus our assumption that $L \notin S$ is true is incorrect. Therefore $L \in S$ because (Set Theory and Logic) of the "Law of the Excluded Middle:" the statement "$L \in S''$ has to be either True or False.

(02) **Theorem:** *If a sequence of real numbers converges, then it is bounded.*

Proof: Suppose $\lim_{n\to\infty} x_n = L$. Take $\epsilon = 1$ in the definition of sequential limit. Then there exists $K \in \mathbb{N}$ such that $n \ge K$ implies that $|x_n - L| < 1$. Thus $n \ge K \Rightarrow |x_n| \le |L| + 1$ (why?!). Since the set $\{|x_0|, \ldots, |x_K|\}$ is finite, it is bounded by its largest element. Hence for all $n \in \mathbb{N}$, $|x_n| \le \max\{|x_0|, \ldots, |x_K|, |L|+1\}$. This is the definition of "$\{x_n\}$ is bounded."

(03) **Theorem:** *Limits of sequences of real numbers are unique.*

Proof: Suppose that $\lim_{n\to\infty} x_n = L_1$ and $\lim_{n\to\infty} x_n = L_2$ and that $L_1 \ne L_2$. Let $\epsilon := |L_1 - L_2| > 0$. There are natural numbers K_1 and K_2 such that whenever $n \ge K_1$, $|x_n - L_1| < \epsilon/2$, and, whenever $n \ge K_2$, $|x_n - L_2| < \epsilon/2$. Therefore, if $n \ge \max\{K_1, K_2\}$,

$$\epsilon = |L_1 - L_2| \le |L_1 - x_n| + |x_n - L_2| < (\epsilon/2) + (\epsilon/2) = \epsilon, \quad \text{a contradiction.}$$

(04) **Exercise:** *Prove that for all real numbers* x *and* y, $|x \pm y| \ge |x| - |y|$.

(05) **Theorem:** *If* $\{x_n\}$ *converges (i.e., has a limit) then* $\{x_n\}$ *is bounded (i.e., there exists* A *such that* $|x_n| \le A$ *for all* n*).*

Proof: Suppose that $\lim_{n\to\infty} x_n = L$. Choose $\epsilon = 1$. Then there exists K so large that if $n \ge K$, $|x_n - L| < 1 = \epsilon$. Therefore for $n \ge K$, $|x_n| = |x_n - L + L| \le |x_n - L| + |L| < 1 + |L|$. Since every n is either less than K or at least K,

$$|x_n| \le \max\{|x_0|, |x_1|, \ldots, |x_{K-1}|, 1 + |L|\}.$$

Limit Theorems

(06) **Theorem:** *If* $\lim_{n\to\infty} x_n = L$ *and* $\lim_{n\to\infty} y_n = M$ *then* $\lim_{n\to\infty} x_n y_n = LM$.

Proof: By (05) there exists A such that $|y_n| \le A$ for all n. Therefore

(07) $$|x_n y_n - LM| = |x_n y_n - L y_n + L y_n - LM| \le |x_n - L||y_n| + |L||y_n - M| \le |x_n - L| A + |L||y_n - M|.$$

We next obtain K_1 and K_2 from the existence of the limits of $\{x_n\}$ amd $\{y_n\}$, corresponding to $\dfrac{\epsilon}{2(A+1)}$ and $\dfrac{\epsilon}{2(|L|+1)}$ respectively. Then, if $n \ge K := \max\{K_1, K_2\}$ we have, from (07), $|x_n y_n - LM| < \epsilon$.

(08) **Exercises:** *State and prove the Limit Theorems for sums and constant multiples.*