

Theorem (The Simple Vitali Lemma): *There exists a positive number β depending only on n such that for all sets $E \subseteq \mathbb{R}^n$ with $|E|_e < \infty$, if K is a non-empty family of cubes Q that covers E , there exists a finite disjoint subfamily $\{\tilde{Q}_1, \dots, \tilde{Q}_M\}$ of K such that (*) $\beta|E|_e < \sum_{j=1}^M |\tilde{Q}_j|$.*

Proof: There are three cases, two of which are easy. The third case has subcases and sub-subcases, and is based on a simple selection process.

Case 0: $|E|_e = 0$. We choose any cube $\tilde{Q}_0 \in K$. Then for every $\beta > 0$, (*) holds for the family $\{\tilde{Q}_0\}$. We note that the lemma does not demand that the family sought covers any part of E !

Case 1: $|E|_e > 0$ and $\sup_{Q \in K} |Q| = \infty$. Then there exists a sequence $Q_k \in K$ such that $|Q_k| \uparrow \infty$. Given any $\beta > 0$, there is a first M such that $|Q_M| > \beta|E|_e$. Then (*) holds for the family $\{\tilde{Q}_M\}$.

Case 2: $|E|_e > 0$ and $t := \sup_{Q \in K} |Q| < \infty$. As expressed by the authors, the key idea is to select cubes as large as we can. Since there may not be a cube with largest measure, we will select one whose measure is at least half the supremum we work with. Thus we begin by selecting a cube $\tilde{Q}_1 \in K$ such that $|\tilde{Q}_1| > t/2$.

In our work on this Case, we will need a *finite* cover of E , but not by sets from K . The covering sets will be expansions about their centers of cubes in K . More precisely, given any cube Q we let c_Q denote its center, and we can then write $Q = c_Q + eQ_0$, where Q_0 is the cube with center zero and edge one. We will then write $Q^* := c_Q + 5eQ_0$, the cube with the same center as Q but with five times the edge length. Since $|Q^*| = 5^n|Q|$, we will be able to get (*) once we know that, starting with a certain finite family of cubes Q from K , the family of related cubes Q^* covers E . The important fact here is that if the edge of Q is greater than half the edge of a cube \hat{Q} and the two cubes meet, then $Q^* \supseteq \hat{Q}$.

Exercise: Verify the preceding sentence's claim.

In this Case we will also use the Recursion Theorem to set up a recursive sequence of finite subcollections of K . Let us define \mathcal{F} to consist of the empty subfamily of K and all finite subfamilies $F \subseteq K$ with this property:

$$\text{for every } Q \in K, \text{ there exists } \tilde{Q} \in F \text{ such that } |\tilde{Q}| > \frac{1}{2}|Q|.$$

Since $\{\tilde{Q}_1\} \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$, \mathcal{F} is not empty, and has at least one non-empty member. We will let \tilde{Q} denote a typical cube in a non-empty member F of \mathcal{F} . We now define a function $F \mapsto K_F \subseteq K$ as follows:

$$K_F := \begin{cases} \emptyset \ (\in 2^K), & \text{if } F = \emptyset; \\ \{Q \in K : Q \cap \tilde{Q} = \emptyset\}, & \text{if } F \neq \emptyset. \end{cases}$$

We need to notice that K_F can be empty even if F is not, in case every Q in K meets some $\tilde{Q} \in F$. We define K'_F to be the complement in K of K_F , so K'_F is the collection of all members Q of K that meet some $\tilde{Q} \in F$. Next, we define $t_F := \sup_{Q \in K'_F} |Q|$. Finally, we define Q_F , if $K_F \neq \emptyset$, to be a cube \hat{Q} in K_F that satisfies the condition $|\hat{Q}| > t_F/2$. Now we are prepared to define the function $h : \mathcal{F} \rightarrow \mathcal{F}$ that we will use in our application of the Recursion Theorem:

$$h(F) := \begin{cases} \emptyset, & \text{if } F = \emptyset; \\ \emptyset, & \text{if } K_F = \emptyset; \\ F \cup \{Q_F\}, & \text{if } K_F \neq \emptyset. \end{cases}$$

We choose $\{\tilde{Q}_1\}$ as our element of \mathcal{F} . The Recursion Theorem yields a unique sequence $\{F_n\}$ in \mathcal{F} such that $F_1 = \{\tilde{Q}_1\}$ and, for all $m \in \mathbb{Z}^+$, $F_{m+1} = h(F_m)$. If $F_{m+1} \neq \emptyset$ then $F_{m+1} \supseteq F_m$. By construction, F_m has m elements if $F_m \neq \emptyset$, and each F_m is a disjoint collection of cubes in K . Thus $0 < t_{F_m} \downarrow$.

We have two Subcases to consider:

Subcase 2a: there exists $\delta > 0$ such that $t_{F_m} \geq \delta$ for all m . Subcase 2b: $t_{F_m} \rightarrow 0$.

In subcase 2a, no F_m is empty. Then, for all m , $\tilde{Q} \in F_m \Rightarrow |\tilde{Q}| > \delta/2$. Hence $\sum_{\tilde{Q} \in F_m} |\tilde{Q}| \rightarrow \infty$ so for all $\beta > 0$,

there is a first m , call it M , such that $(*)$ holds.

Subcase 2b has two Sub-subcases: Sub-subcase 2ai: some F_m is empty. Sub-subcase 2aii: $0 < t_{F_m} \rightarrow 0$.

In 2ai, there is a first m such that F_m is empty; call it $M+1$. Then F_M is not empty, but now every $Q \in K$ meets some \tilde{Q} in F_M . Therefore $K = K'_{F_M}$, so every $Q \in K$ is contained in some \tilde{Q}^* , where $\tilde{Q} \in F_M$. Since K covers E , so do the \tilde{Q}^* with $\tilde{Q} \in F_M$. Hence $|E|_e \leq \sum_{\tilde{Q} \in F_M} |\tilde{Q}^*| = 5^n \sum_{\tilde{Q} \in F_M} |\tilde{Q}|$. Here we find that $(*)$ holds

if $0 < \beta \leq 5^{-n}$.

In 2aii, no F_m is empty. If we let Q_m denote the cube in F_m that is not in any preceding F_k , we then obtain a sequence of cubes. We now claim that every $Q \in K$ meets some Q_m , and therefore every $Q \in K$ is contained in some Q_m^* . Otherwise, some cube $Q_\omega \in K$ would have to belong to every one of the K_{F_m} . But then $t_{F_m} \geq |Q_\omega|$, contradicting $t_{F_m} \rightarrow 0$. The claim thus stands. Then

$$E \subseteq \bigcup_m \bigcup_{\tilde{Q} \in F_m} \tilde{Q}, \text{ so } |E|_e \leq \sup_m \sum_{\tilde{Q} \in F_m} |\tilde{Q}^*| = 5^n \sup_m \sum_{\tilde{Q} \in F_m} |\tilde{Q}|.$$

Now, given any β , $0 < \beta < 5^{-n}$, there is a first m , call it M , such that $(*)$ holds in this Sub-subcase.

If we now select $\beta := 5^{-n}/2$, $(*)$ holds in all cases.