

We are given that  $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  is continuous in each variable separately, and we are asked to show that  $f$  is measurable.

**Lemma:** Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous. Then the sequence of continuous piecewise linear functions  $T_n(f)$  determined by  $T_n(f)(k/2^n) := f(k/2^n)$  for  $0 \leq k \leq 2^n$  converges uniformly to  $f$  in  $[0, 1]$ .

*Proof:* Let  $\epsilon > 0$  be given. By uniform continuity there exists  $\delta > 0$  such that  $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$ . Then for all  $n$  such that  $1/2^n < \delta$  we have, for  $x_k := k/2^n \leq x \leq x_{k+1} = (k+1)/2^n$ ,

$$\begin{aligned} |f(x) - T_n(f)(x)| &= |f(x) - 2^n(x_{k+1} - x)f(k/2^n) - 2^n(x - x_k)f((k+1)/2^n)| \\ &= |2^n(x_{k+1} - x)(f(x) - f(k/2^n)) + 2^n(x - x_k)(f(x) - f((k+1)/2^n))| \\ &\leq 2^n(x_{k+1} - x)\epsilon + 2^n(x - x_k)\epsilon = \epsilon, \end{aligned}$$

as desired.

We now define  $f_n(x, y) := T_n(f(\bullet, y))(x)$ . That is, we define the function  $f(\bullet, y) : [0, 1] \rightarrow \mathbb{R}$  to be  $f(x, y)$ ; we treat  $y$  as a “parameter” and apply the Lemma to  $f(\bullet, y)$ . This function is continuous in both variables, in the usual sense, hence is measurable. As  $n \rightarrow \infty$ ,  $f_n \rightarrow f$  pointwise, tho not necessarily uniformly, because the functions  $f(\bullet, y)$  may not have a common modulus of uniform continuity. In any case, the function  $f$  is the pointwise limit of a sequence of measurable functions, so  $f$  is measurable.

**Remark:** This solution completely bypasses the fascinating problem of understanding the structure of the set where  $\{f > \alpha\}$ !