Special Problem 7: Due August 3

§**20.1** #4.

(a) Find a simple expression for the function $f(x) = \sum_{n=0}^{\infty} x^2/(1+x^2)^n$ when $x \neq 0$. What is $f(0)$? Does $f(x)$ have any discontinuities?
(b) If $s_n(x)$ is the sum of the first n terms of the series, show that $0 < f(x) - s_n(x) \leq \dfrac{1}{(1+\delta^2)^{n-1}}$ if $x^2 \geq \delta^2 > 0$. What do you conclude about uniform convergence?
(c) Make a sketch showing the appearance of the curve $y = s_n(x)$ for a very large value of n. Is the series uniformly convergent when $|x| \leq \delta$?

(a): $x \neq 0$ and $0 < 1/(1+x^2) < 1$ so $f(x)$ is, when $x \neq 0$, x^2 times a geometric series. Thus

$$f(x) = x^2 \sum_{n=0}^{\infty} \left(\frac{1}{1+x^2}\right)^n = x^2 \frac{1}{1 - \frac{1}{1+x^2}} = x^2 \frac{1}{\frac{1+x^2-1}{1+x^2}} = 1 + x^2, \quad \text{if } x \neq 0.$$

But $f(0) = 0$ because every term in the series for $f(x)$ is zero. Thus $f(x)$ is discontinuous at $x = 0$.
(b): The first n terms are the ones with exponents running from 0 thru $n-1$. You're supposed to *know* the formula for the partial sums of a geometric series. Thus if $x \neq 0$,

$$f(x) - s_n(x) = 1 + x^2 - x^2 \frac{1 - \left(\frac{1}{1+x^2}\right)^n}{1 - \frac{1}{1+x^2}} = 1 + x^2 - x^2(1+x^2)\frac{1 - \left(\frac{1}{1+x^2}\right)^n}{x^2} = \left(\frac{1}{1+x^2}\right)^{n-1} \leq \left(\frac{1}{1+\delta^2}\right)^{n-1}$$

if $|x| \geq \delta$. The difference above is positive because every term of our series is positive, so the series sum is greater that each partial sum. Thus $|f(x) - s_n(x)| \leq \left(\frac{1}{1+\delta^2}\right)^{n-1} \to 0$ as $n \to \infty$. By our necessary-and-sufficient condition for uniform convergence, we have uniform convergence when $|x| \geq \delta$.
(c): The picture will (as one of you said!) look like a giant wine glass with a very skinny stem. But we can answer the uniform convergence without the picture. If it *were* true that $s_n(x) \to f(x)$ uniformly, then (since each $s_n(x)$ is continuous (the formula in (b) displays $s_n(x)$), the limit function, $f(x)$ would be continuous. It is not, by (a), so the convergence is not uniform.

Quiz 6 # 3, second part: State the *Chain Rule.* Context: $\mathbb{R}^n \to \mathbb{R}^n$.
Find the derivative of $f \circ g(x)$ when $g(x) := (x_1^2, x_2^2, \ldots, x_n^2)$ and $f(y) := (y_n + y_1, y_1 + y_2, \ldots, y_{n-1} + y_n)$.

We need to find $f'(y)$ and $g'(x)$, then put them together.

$f_i(y) = y_{i-1} + y_i$, where by y_0 we mean y_n. Then $\frac{\partial f_i}{\partial y_k} = \delta_{i-1,\,j} + \delta_{i,\,j}$, where by $\delta_{0,\,j}$ we mean $\delta_{n,\,j}$. Therefore we get, for $f'(y)$,

$$f'(y) = (\delta_{ij})_{n\times n} + J = I + J, \quad \text{where } J = \begin{pmatrix} 0_{1\times(n-1)} & 1_{1\times 1} \\ I_{(n-1)\times(n-1)} & 0_{(n-1)\times 1} \end{pmatrix} \quad \text{and}$$

$I_{(n-1)\times(n-1)}$, *inside* the matrix J is the diagonal just below the main diagonal.

The matrix J is written in *block form.* Block form is useful because if A and B are written in block form and we treat the blocks as objects to be multiplied and added and the blocks have shapes that allow them to be multiplied, the multiplication rule is the same as the ordinary rule for multiplying matrices: "dot and add." However the results of our "dot products" are matrices.

Since $g_k = x_k^2$, $\frac{\partial g_k}{\partial x_j} = 2\delta_{k,\,j}$, so that

$$g'(x) = \text{Diag}\,(2x), \quad \text{meaning the matrix } D \text{ whose diagonal entries } d_{ii} = 2x_i \text{ and } d_{ij} = 0 \text{ if } i \neq j.$$

Then

$$(f\circ g)'(x) = (I+J)\text{Diag}\,(2x) = 2\begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 & x_n \\ x_1 & x_2 & 0 & \cdots & 0 & 0 \\ 0 & x_2 & x_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & x_{n-1} & x_n \end{pmatrix}.$$

§**6.4 # 3(b):** Show that $\sum_{k=1}^n x_k d(x_k/r) = 0$.

First step: Calculate $d(x_k/r)$. In general, $df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j$, where each dx_j is a number; df is then a *linear* function of the vector dx (whose jth component is dx_j). The quantities $\frac{\partial f}{\partial x_j}$ depend on the point x at which they are evaluated.

Thus

$$d(x_k/r) = \sum_{j=1}^n \frac{\partial(x_k/r)}{\partial x_j} dx_j = \sum_{j=1}^n \left[\frac{\delta_{k,j}}{r} - \frac{x_k}{r^2}\frac{\partial r}{\partial x_j}\right] dx_j = \sum_{j=1}^n \left[\frac{\delta_{k,j}}{r} - \frac{x_k}{r^2}\frac{x_j}{r}\right] dx_j = \sum_{j=1}^n \left[\frac{\delta_{k,j}}{r} - \frac{x_k x_j}{r^3}\right] dx_j.$$

Then we multiply by x_k and add, then *change the order of summation:*

$$\begin{aligned}\sum_{k=1}^n x_k d(x_k/r) &= \sum_{k=1}^n\sum_{j=1}^n \left[\frac{x_k\delta_{k,j}}{r} - \frac{x_k^2 x_j}{r^3}\right] dx_j = \sum_{j=1}^n\sum_{k=1}^n \left[\frac{x_k\delta_{k,j}}{r} - \frac{x_k^2 x_j}{r^3}\right] dx_j \\ &= \sum_{j=1}^n \left[\frac{x_j}{r} - \sum_{k=1}^n \frac{x_k^2 x_j}{r^3}\right] dx_j = \sum_{j=1}^n \left[\frac{x_j}{r} - \frac{r^2 x_j}{r^3}\right] dx_j = \sum_{j=1}^n 0 dx_j = 0.\end{aligned}$$

Notice that, in the sum over k that involves $\delta_{k,j}$, only one term survives: the one with $k=j$. In the other sum, the x_k's add up to r^2 and this cancels part of the denominator.

§**6.52 # 7:** If $u = \phi(x-ct) + \psi(x+ct)$, show that $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$.

First we find $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$. To do so we use the Chain Rule for functions of one variable:

$$\frac{\partial u}{\partial t} = -c\phi'(x-ct) + c\psi'(x+ct) \quad\text{and}\quad \frac{\partial u}{\partial x} = \phi'(x-ct) + \psi'(x+ct).$$

We differentiate each of these once more:

$$\frac{\partial^2 u}{\partial t^2} = +c^2\phi''(x-ct) + c^2\psi''(x+ct) \quad\text{and}\quad \frac{\partial^2 u}{\partial x^2} = \phi''(x-ct) + \psi''(x+ct).$$

Examining the two formulas (on their right-hand sides) we see that if we multiply the second one by c^2 the result is the first one, which is what we had to show. **Assignment 2,** (2) §**2.8** #8, p 84: Reason for solution request: "I had no clue how to begin!"

We are given real numbers a_n, $n \ge 1$, such that $(2-a_n)a_{n+1} = 1$, all n.
We notice: $a_n \ne 2$, all n.
(a) Show that $\lim\limits_{n\to\infty} a_n$ exists. We are told to consider two cases:
(a1): Assume $a_n < 2$, all n. Then (hope the a_n's increase)

$$(*)\qquad a_n - a_{n+1} = a_n - \frac{1}{2-a_n} = \frac{2a_n - a_n^2 - 1}{2-a_n} = -\frac{(a_n-1)^2}{2-a_n} \le 0, \quad\text{so } \{a_n\} \text{ increases.}$$

Since $a_n < 2$ for all n, $\{a_n\}$ is increasing and bounded above, so $\lim\limits_{n\to\infty} a_n$ exists in this case.

(a2): Assume $a_n > 2$, *some* n. We obtain such an n and denote it by N. Then $a_{N+1} = \dfrac{1}{2-a_N} < 0$ (so we can't hope that all the later a_n's increase!). With no other idea, we look at $a_{N+2} = \dfrac{1}{2-a_{N+1}} > 0$ and, because the denominator exceeds 2, $a_{N+2} = \dfrac{1}{2-a_{N+1}} < \dfrac{1}{2}$. Thus $2 - a_{N+2} > \frac{3}{2}$. Hence $0 < a_{N+3} = \dfrac{1}{2-a_{N+2}} < \dfrac{2}{3}$. *Maybe* it's true that: *for all* $n > N$, $a_n < 1$. We can try to prove that statement by an induction starting at $N+1$.
Basis Step: $a_{N+1} < 0 < 1$.
Induction Step: Prove that *If* $n \geq N+1$ *and* $a_n < 1$ *then* $a_{n+1} < 1$.
Proof: $a_n < 1$ implies that $2 - a_n > 1$. Thus $a_{n+1} = \dfrac{1}{2-a_n} < 1$. This completes the proof of the statement.

Now we go back to the work in $(*)$ and we see that it's true now for all $n > N$, so the sequence $\{a_n\}_{n=N}^{\infty}$ is increasing and bounded above by 1 now so it has a limit. Since "having a limit" is not influenced by the terms a_n with $n \leq N$, part (a) is proved.
Comment: How do we get to the "Maybe?" I just calculated a few more of the a_n's and guessed that would happen.

(b) To find the limit (call it L) we use something we know about sequences: *If* $\lim_{n\to\infty} a_n = L$ *then* $\lim_{n\to\infty} a_{n+1} = L$, so we can use the product Limit Theorem to see that $1 = (2-a_n)a_{n+1} \to (2-L)L$. That is, $2L - L^2 = 1$, so $L^2 - 2L + 1 = 0 = (L-1)^2$, so $L = 1$.