From **Test 1:**

(3) [10] Find the *projection* of $f(t) = \sin^2 t$ on $\operatorname{span}(\{e_n : 0 \le n \le 3\}) \subseteq L^2(0, 2\pi)$. Here, $e_n(t) = e^{int}$.

Let $X := \operatorname{span}(\{e_n : 0 \le n \le 3\})$. Since finite-domensional spaces are closed, X is a closed subspace, and $\mathcal{O} := \{e_n : 0 \le n \le 3\}$ is an orthonormal basis of X. The "general" formula for $P_X y$ is

$$P_X y = \sum_{v \in \mathcal{O}} \langle y,\, v\rangle\, v, \quad \text{so in our case}$$

$$P_X f = \sum_{n=0}^{3} \langle f,\, e_n\rangle\, e_n \quad \text{and} \quad \langle f,\, e_n\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 t\, e^{-int}\, dt.$$

Since f is even, the sine parts of the functions e^{-int} contribute zero to the inner product. The trigonometric identity $2\sin^2 t = 1 - \cos 2t$ then shows that $\langle f,\, e_1\rangle = 0 = \langle f,\, e_3\rangle$. All we have to *compute* are

$$\langle f,\, e_0\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\cos 2t}{2}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\, dt = \frac{1}{2} \quad \text{and}$$

$$\langle f,\, e_2\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\cos 2t}{2}\, e^{-i2t}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{-\cos^2 2t}{2}\, dt = \frac{-1}{4\pi}\int_{-2\pi}^{2\pi} \cos^2 t\, dt/2, \qquad t \mapsto t/2;$$

now we use the trigonometric identity $2\cos^2 t = 1 + \cos 2t$ and continue:

$$\langle f,\, e_2\rangle = \frac{-1}{8\pi}\int_{-2\pi}^{2\pi} \frac{1+\cos 2t}{2}\, dt = \frac{-1}{8\pi}\frac{4\pi}{2} = -\frac{1}{4}, \quad \text{or} \quad P_X f(t) = \frac{1}{2} - \frac{e^{i2t}}{4}.$$

Remark: $\langle f,\, e_{-2}\rangle = -\frac{1}{4}$ as well, with $\langle f,\, e_{-1}\rangle = 0 = \langle f,\, e_{-3}\rangle$. Thus, were $X = \operatorname{span}(\{e_n : -3 \le n \le 3\})$ we would have had $P_X f = f$, or $\sin^2 t = \frac{1}{2} - \frac{e^{i2t}+e^{-i2t}}{4}$.

From **Test 2:**

(4) [15] Show that the subspace of $L^2(\mathbb{R})$ that consists of all functions in $L^2(\mathbb{R})$ that are zero almost everywhere *outside* the interval $[0, 1]$ is a closed subspace of $L^2(\mathbb{R})$.

The solution is on page 2.

(6) [15] Show that if $f(t) \in L^1(\mathbb{R})$ then $\widehat{f}(\xi)$ is a continuous function of ξ. Clearly identify any Lebesgue Facts that you use.

The solution is given next.

Assignment 8 Due Mar. 30:

#2: Show that if $f(t) \in L^1(\mathbb{R})$ then $\widehat{f}(\xi)$ is a continuous function of ξ. We used this to evaluate $\widehat{\Phi}(0)$ *as an integral.*

$$\widehat{f}(\xi) = \int f(t) e^{-i\xi t}\, dt, \quad \text{so} \quad \widehat{f}(\xi) - \widehat{f}(\xi_o) = \int f(t)\left(e^{-i\xi t} - e^{-i\xi_o t}\right) dt.$$

We will use *Lebesgue's Dominated Convergence Theorem;* we thus need a function $g(t) \in L^1(\mathbb{R})$ such that

$$\left|f(t)\left(e^{-i\xi t} - e^{-i\xi_o t}\right)\right| \le g(t) \ \text{ a.e.}; \ g(t) := 2f(t) \ \text{ works since } \ \left|e^{-i\xi t} - e^{-i\xi_o t}\right| \le 2.$$

We also need to know the limit of $f(t)\left(e^{-i\xi t} - e^{-i\xi_o t}\right)$ as $\xi \to \xi_o$. The limit is zero a.e., since $e^{i\xi t} \to e^{i\xi_o t}$ for every t, as $\xi \to \xi_o$. This does not *exactly* use LDC, becuase LDC is stated in terms of a sequence of functions. We can make it all exact by using the "sequences are good enough" theorem: $F(\xi)$ is continuous at ξ_o if and only if $F(\xi_n) \to F(\xi_o)$ for *every* sequence $\xi_n \to \xi_o$.

Assignment 11 Due Apr. 20:

In these two problems, I had the Box-function context in mind! Nobody asked about it! Thus the answers here are in the Box-function context. In the general context, we have to use $M_2 - M_1 + 1$ instead of 2.

#2: A counting problem. Suppose that $f \in V_J$, and that $f = \sum_{k=K_1+1}^{K_1+2^m} \langle f, \varphi_{Jk} \rangle \varphi_{Jk}$. When we write $f = g_1 + f_1$, where $g_1 \in W_{J-1}$ and $f_1 \in V_{J-1}$, how many coefficients *may* be needed to express g_1 in terms of the $\psi_{J-1,k}$? How many coefficients *may* be needed to express f_1 in terms of the $\varphi_{J-1,k}$?

Each function φ_{Jk} is a Box of width $1/2^J$. As such it is either the left half or the right half of $\varphi_{J-1,k/2}$, where by $k/2$ I mean the *greatest integer* $\leq k/2$. For example, $5/2$ means 2, $6/2$ means 3, $-5/2$ means -3, and so on.

We can reduce everything to two cases:

$$\varphi(2t) = (\varphi(t) + \psi(t))/2 \;\text{ and }\; \varphi(2t-1) = (\varphi(t) - \psi(t))/2.$$

The first gives a "left-half" formula, $\varphi_{10} = \frac{\sqrt{2}}{2}(\varphi_{00}(t) + \psi_{00}(t))$, the second a "right-half" formula, $\varphi_{11} = \frac{\sqrt{2}}{2}(\varphi_{00}(t) - \psi_{00}(t))$. These two can then be translated into the formulas we want:

$$\varphi_{j,2k}(t) = 2^{j/2}\varphi(2^j t - 2k) = 2^{(j-1)/2}(\varphi_{00}(2^{j-1}t - k) + \psi_{00}(2^{j-1}t - k)) = 2^{-1/2}(\varphi_{j-1,k}(t) + \psi_{j-1,k}(t))$$

and

$$\varphi_{j,2k+1}(t) = 2^{j/2}\varphi(2^j t - 2k - 1) = 2^{-1/2}(\varphi_{j-1,k}(t) - \psi_{j-1,k}(t)).$$

In the general context there will be more terms than two on the right-hand sides. In our case though, we'll write

$$f = \sum_{k=K_1+1}^{K_1+2^m} \langle f, \varphi_{Jk} \rangle \varphi_{Jk} =: \sum_{k=0}^{2^m-1} f_k\, \varphi_{J,K_1+1+k} = \sum_{k=0}^{2^{m-1}-1} f_{2k}\, \varphi_{J,K_1+1+2k} + \sum_{k=0}^{2^{m-1}-1} f_{2k+1}\, \varphi_{J,K_1+2+2k}.$$

We avoid messiness here by realizing that in *this* problem we don't want the actual formulas, only the numbers of terms in each. The actual formulas will depend on whether K_1 is even or odd. We notice that when we substitute for each φ_{J,K_1+1+2k} or φ_{J,K_1+2+2k} we get two terms, one using a $2^{-1/2}\varphi_{J-1,k'}$ and the other a $2^{-1/2}\psi_{J-1,k'}$. *Notice that no powers of 2 with a $J-1$ appear!* We then regroup all these terms so that one sum has only φ terms, the other only ψ terms. Moreover, each of the regrouped sums has just 2^{m-1} terms. The sum with the ψ terms we will call g_1, the sum with the φ terms we will call f_1. Each sum has up to 2^{m-1} terms. We then start over, working *only* on f_1, getting $f_1 = g_2 + f_2$, and so on. At step ℓ we have at most $2^{m-\ell}$ coefficients.

#3 (continues #2): If the process started in #2 is continued, so that after N steps we have $f = g_1 + \cdots + g_N + f_N$, how many coefficients *may* be needed? If N is large enough then because we started with a *finite* swath of coefficients, f_N is eventually 0. What is the *first* N such that $f_N = 0$? Why?

In the problem statement I should have asked about the function g_N, not f_N. It is g_N that is eventually zero. From the (extra) stuff in the #2 answer, we see that when we get to step $m+1$, g_{m+1} uses zero coefficients. But f_{m+1} will be nonzero if the integral of f is not zero. Its integral will be "spread out" over the width of a Box function large enough to enclose the interval on which f started.

From **Test 2:**
(4) [15] Show that the subspace of $L^2(\mathbb{R})$ that consists of all functions in $L^2(\mathbb{R})$ that are zero almost everywhere *outside* the interval $[0, 1]$ is a closed subspace of $L^2(\mathbb{R})$.

(Much more detail than was expected:) Let V denote the given subspace. Suppose that $f_n \in V$ and $f_n \to f$ in $L^2(\mathbb{R})$ as $n \to \infty$. Then because $f_n(t) = 0$ a.e. if $t < 0$ or $t > 1$,

$$(*) \qquad \begin{aligned} \|f_n - f\|^2 &= \int_{-\infty}^{0} |f_n(t) - f(t)|^2\,dt + \int_0^1 |f_n(t) - f(t)|^2\,dt + \int_1^\infty |f_n(t) - f(t)|^2\,dt \\ &= \int_{-\infty}^{0} |f(t)|^2\,dt + \int_0^1 |f_n(t) - f(t)|^2\,dt + \int_1^\infty |f(t)|^2\,dt. \end{aligned}$$

Since $\|f_n - f\|^2 \to 0$ and since $\int_0^1 |f_n(t) - f(t)|^2\,dt \le \|f_n - f\|^2$, the left-hand side in the second line of $(*)$ tends to zero and the right-hand side tends to $\int_{-\infty}^0 |f(t)|^2\,dt + \int_1^\infty |f(t)|^2\,dt$, so that $\int_{-\infty}^0 |f(t)|^2\,dt + \int_1^\infty |f(t)|^2\,dt = 0$. Finally, we apply (18) in "Lebesgue Facts:"

(18) If $f(x) \ge 0$ and $f(x)$ is measurable and $\int f(x)\,dx = 0$, then $f(x) = 0$ a.e.

Therefore $|f(t)|^2 = 0$ a.e. outside $[0, 1]$, hence $f(t) = 0$ a.e. outside $[0, 1]$. Thus $f \in V$, so we have shown that V is a closed subspace of $L^2(\mathbb{R})$.

Assignment 7 Due Mar. 23:

#1: Supposing that h_0, h_1, h_2, h_3 are real and that all other h_n are zero, find a formula for h_1 in terms of h_0 that allows you to find all solutions, for such h_n's, of $(h1)$ – $(h4)$. Your solutions should include the ones with just two non-zero h_n's among h_0, h_1, h_2 and h_3.

From $(h2)$, $\sum_{n\in\mathbb{Z}} h_n = \sqrt{2}$, and $(h3)$, $\sum_{n\in\mathbb{Z}}(-1)^n h_n = 0$, we get the equations

$$\begin{array}{c} h_0 + h_1 + h_2 + h_3 = \sqrt{2} \\ h_0 - h_1 + h_2 - h_3 = 0 \end{array} \quad \text{and these reduce to} \quad \begin{array}{c} h_0 + h_2 = \sqrt{2}/2 \\ h_1 + h_3 = \sqrt{2}/2 \end{array}$$

by first adding the equations, then subtracting them. The equations then allow us to note that $h_2 = \sqrt{2}/2 - h_0$ and $h_3 = \sqrt{2}/2 - h_1$. Hence we only have to find h_0 and h_1. Next we look at $(h4)$. It turns out that we only have to consider the cases $k = 0$ and $k = 1$ in $(h4)$. As review, you should now write those two cases down. Then substitute in the values for h_2 and h_3 in terms of h_0 and h_1. in *both* cases we get the same equation! It is:

$$h_0^2 + h_1^2 = \frac{\sqrt{2}}{2}(h_0 + h_1), \quad \text{or} \quad h_1^2 - \frac{\sqrt{2}}{2}h_1 + \left(h_0^2 - \frac{\sqrt{2}}{2}h_0\right) = 0.$$

This last is a quadratic equation with the form $Ah_1^2 + Bh_1 + C = 0$, where $A = 1$, $B = -\frac{\sqrt{2}}{2}$ and $C = h_0^2 - \frac{\sqrt{2}}{2}h_0$. The coefficient C contains a *parameter* h_0. This parameter can only take values that ensure that $B^2 - 4AC \ge 0$. When you write out (review!) $B^2 - 4AC$ in terms of h_0 you find that $B^2 - 4AC$ is a quadratic with variable h_0 that has a *negative* highest coefficient. Therefore the graph of that quadratic points *down*, so the quadratic can only be non-negative if h_0 lies between the two roots of the quadratic. This leads to the (messy!) answer

$$h_1 = \frac{\pm\sqrt{\frac{1}{2} - 4\left(h_0^2 - \frac{\sqrt{2}}{2}h_0\right)}}{2}, \quad \text{where} \quad \frac{\sqrt{2}}{4} - \frac{1}{2} \le h_0 \le \frac{\sqrt{2}}{4} + \frac{1}{2}.$$

An alternate solution

The quadratic $h_0^2 + h_1^2 = \frac{\sqrt{2}}{2}(h_0 + h_1)$ can be rewritten, with the help of completing the squares and normalizing:

$$\left(2h_0 - \frac{\sqrt{2}}{2}\right)^2 + \left(2h_1 - \frac{\sqrt{2}}{2}\right)^2 = 1.$$

We then recognize that this allows us to use an *angle* as our parameter: set $2h_0 - \frac{\sqrt{2}}{2} = \cos\theta$ and $2h_1 - \frac{\sqrt{2}}{2} = \sin\theta$.

Assignment 10 Due Apr. 13: The only problem on this assignment: Prove this statement, that we have used, and will be using again soon: *If H is a Hilbert space and $S \subseteq H$ then span(S) is dense in H if and only if it is true that: $y \perp S \iff y = 0$.* This is in part a **review** problem: you'll probably have to review the Hilbert Space notes and definitions such as "dense..." Questions are welcome!

We were given a subset S of H. We have two things to prove:
(1) If span(S) is dense in H then for all $y \in H$, $y \perp S \Rightarrow y = 0$.
(2) If, for all $y \in H$, $y \perp S \Rightarrow y = 0$ then span(S) is dense in H.
We notice that the "$y = 0 \Rightarrow y \perp S$" part can be ignored.

In (1) we get to *assume* that $\operatorname{span}(S)$ is dense in H. If we then assume that $y \perp S$ we have to deduce that $y = 0$. Here's how: we know that $\operatorname{span}(S)$ is dense in H, so there are vectors $s_n \in \operatorname{span}(S)$ such that $\|s_n - y\| \to 0$. Therefore $\|y\|^2 = \langle y,\, y\rangle = \langle y - s_n + s_n,\, y\rangle = \langle y - s_n,\, y\rangle + \langle s_n,\, y\rangle = \langle y - s_n,\, y\rangle$ because for each n, $s_n \perp y$. Thus by the Schwarz Inequality $\|y\|^2 = \langle y - s_n,\, y\rangle \le \|y - s_n\|\,\|y\| \to 0$ as $n \to \infty$. Thus $y = 0$ (why?: review).

In (2) we get to *assume* that, for all $y \in H$, $y \perp S \Rightarrow y = 0$. We define $Y := \overline{\operatorname{span}(S)}$. Then Y is a closed subspace of H. We want to show that *every* $x \in H$ is in Y. One way to do so is to show that for all $x \in H$, $x - P_Y x = 0$ (why?: review). Let $y := x - P_Y x$. Then $y \in Y^\perp$ (why?: review). By our hypothesis (assumption) it follows that $y = 0$. Thus $H \subseteq \overline{\operatorname{span}(S)}$. This completes the proof.

Assignment 8 Due Mar. 30:

#1: Show that the set of all $f \in L^2(\mathbb{R})$ such that $f(t) = 0$ a.e. in $(M_1,\, M_2)$ is a closed subspace of $L^2(\mathbb{R})$. We used this when we proved that our $\Phi(t)$ has compact support contained in $[M_1,\, M_2]$.

Let's write $V_{M_1M_2}$ to denote the set of functions in $L^2(\mathbb{R})$ that are zero a.e. in $[M_1,\, M_2]$. Since these functions are zero a.e. in $[M_1,\, M_2]$, so is the sum of two of them and every scalar multiple of each of them. Thus $V_{M_1M_2}$ is closed under addition and scalar multiplication, so $V_{M_1M_2}$ is a subspace (in the linear-algebra sense). It remains to show that $V_{M_1M_2}$ is closed with respect to Cauchy sequences. Thus, if each $f_n \in V_{M_1M_2}$ and $\|f_n - f\| \to 0$, where $f \in L^2(\mathbb{R})$, as $n \to \infty$, we have to show that $f \in V_{M_1M_2}$. We can write

$$\begin{aligned}\|f_n - f\|^2 &= \int_{-\infty}^{M_1} |f_n(t) - f(t)|^2\,dt + \int_{M_1}^{M_2} |f_n(t) - f(t)|^2\,dt + \int_{M_2}^{\infty} |f_n(t) - f(t)|^2\,dt \\ &= \int_{-\infty}^{M_1} |f_n(t) - f(t)|^2\,dt + \int_{M_1}^{M_2} |f(t)|^2\,dt + \int_{M_2}^{\infty} |f_n(t) - f(t)|^2\,dt \\ &\to \int_{M_1}^{M_2} |f(t)|^2\,dt = 0 \quad \text{(the first and third terms are each, at most, } \|f_n - f\|^2\text{)},\end{aligned}$$

so $f(t) = 0$ a.e. in $[M_1,\, M_2]$. Thus $f \in V_{M_1M_2}$. The proof is done.

#3: Suppose we want h_n satisfying $(h4)$, with $h_n = 0$ unless $0 \le n \le N-1$, with $h_0 \ne 0 \ne h_{N-1}$. Show that if $(h4)$ is true for our h_n then N has to be even. Show that if $(h4)$ is true for some $k > 0$ then $(h4)$ is true for $-k$ as well. Find the number of equations that $(h4)$ produces. Hint: For $k>0$, verify that $(h4)$ becomes $\sum_{n=2k}^{N-1} h_n\overline{h_{n-2k}}=0$.

To show that N must be even, suppose that (contrawise) N is odd. Then $N-1$ is even. We can choose any k to use in $(h4)$ so we choose k so that when $n = N-1$, $n - 2k = 0$. That is, $k = (N-1)/2$. Then for $0 \le n < N-1$, $n - 2k = n - (N-1) < N - 1 - (N-1) = 0$. This means that the subscript of h_{n-2k} is negative, so that $h_{n-2k} = 0$ for $0 \le n < N-1$. Thus

$$\sum_{n=0}^{N-1} h_n\overline{h_{n-2k}} = \sum_{n=N-1}^{N-1} h_n\overline{h_{n-2k}} = h_{N-1}\overline{h_0} \ne 0, \quad \text{by hypothesis. This violates } (h4).$$

Hence N could not have been odd: N must be even.

To show that if $(h4)$ is true for some $k > 0$ then $(h4)$ is true for $-k$ as well, we assume that $k > 0$ and $\sum_{n=0}^{N-1} h_n\overline{h_{n-2k}} = 0$. Then we look at

$$(8\#3-1) \qquad \sum_{n=0}^{N-1} h_n\overline{h_{n-2(-k)}} = \sum_{n=0}^{N-1} h_n\overline{h_{n+2k}}.$$

We now make $n \mapsto n - 2k$ and rewrite the sum on the right:

$$(8\#3-2) \qquad \sum_{n=0}^{n=N-1} h_n\overline{h_{n+2k}} = \sum_{n-2k=0}^{n-2k=N-1} h_{n-2k}\overline{h_{n-2k+2k}} = \sum_{n=2k}^{N-1+2k} h_{n-2k}\overline{h_n} = \sum_{n=2k}^{N-1} h_{n-2k}\overline{h_n};$$

the $+2k$ disappears from the upper limit of summation because $h_n = 0$ if $n > N-1$. We combine $(8\#3-1)$ and $(8\#3-2)$ and get

$$(8\#3-3) \qquad \sum_{n=0}^{N-1} h_n \overline{h_{n-2(-k)}} = \sum_{n=2k}^{N-1} h_{n-2k} \overline{h_n} = \overline{\sum_{n=2k}^{N-1} \overline{h_{n-2k}} h_n} = \overline{\sum_{n=2k}^{N-1} h_n \overline{h_{n-2k}}} = \overline{0} = 0,$$

so that $(8\#3-3)$ becomes $(h4)$ with k replaced by $-k$.

Finally, we have to count the number of equations in $(h4)$ that have to be checked. We know now that we only need to work with $k = 0$ and with positive k. Now we look at $k = N/2$ (since N must be even) and put this into $(h4)$. When $n = N-1$, $n - 2k = N - 1 - 2k = N - 1 - N < 0$, so the sum in $(h4)$ will be zero "automatically" for $k \geq N/2$. Thus we only have to check for $0 \leq k < N/2$, giving us a total of $N/2$ equations in $(h4)$ to check.

Special Problem 3: Due Mar. 4

Show that, with our example of V_0 given in terms of the Box function, $\overline{\bigcup_j V_j} = L^2(\mathbb{R})$.

We use the statement *The span of a subset S of H is dense in H if and only if $y \perp S$ implies $y = 0$*, found in the Hilbert space notes between (13) and (13.1). We thus want to show that if $g \in L^2(\mathbb{R})$ and $g \perp \bigcup_j V_j$ then $g = 0$. But if $g \perp \bigcup_j V_j$ then $g \perp 2^{j/2}B(2^j t - k)$ for all j and k in $\mathbb{Z}$ because all the functions $B_{jk}(t) := 2^{j/2}B(2^j t - k)$ are in V_j. Now $g \perp B_{jk}$ is the same as $\langle g, B_{jk}\rangle = 0$ and $B_{jk}(t) = 2^{j/2}$ if $\dfrac{k}{2^j} < t < \dfrac{k+1}{2^j}$ and $B_{jk}(t) = 0$ otherwise. Hence $0 = \langle g, B_{jk}\rangle$ means that the average value of g over $[k/2^j, (k+1)/2^j]$ is zero for all j and k in $\mathbb{Z}$. Since every t for which Lebesgue's Differentiation Theorem is true lies in such an interval, whose lengths tend to zero as $j \to \infty$, the function g has to be zero a.e. Is this solution too brief?

(4) [20] State the *Schwarz Inequality completely.* Use it to show that if $f \in L^2(\mathbb{R})$ then $\displaystyle\int_a^b |f(x)|\,dx < \infty$ if $-\infty < a < b < +\infty$. You need to create a function $g(t)$ so that $\int_a^b |f(x)|\,dx \leq |\langle f, g\rangle|$.

The statement is (5) in the Hilbert space notes. To create the desired g we notice that if $f(x)\overline{g(x)} = |f(x)|$ when $a < x < b$ and $f(x)\overline{g(x)} = 0$ otherwise, then actually $\int_a^b |f(x)|\,dx = |\langle f, g\rangle|$. We know that every complex number $z \neq 0$ can be written in "polar form:" $z = |z|e^{i\theta}$ for some real θ. Thus $f(x) = |f(x)|e^{i\theta(x)}$ if $f(x) \neq 0$, and we define $\theta(x) := 0$ if $f(x) = 0$. We then define $g(x) := e^{i\theta(x)}$ $a < x < b$ and $g(x) := 0$ otherwise. Then $\|g\|^2 = \int_a^b 1\,dx = b - a$ so

$$\int_a^b |f(x)|\,dx = |\langle f, g\rangle| \leq \|f\|\,\|g\| = \|f\|\sqrt{b-a} < \infty.$$

(8) [15] Find the transformation $f(t) \mapsto Tf(x) = \kappa f(\lambda x - \mu)$ that transforms functions of t defined on $I := (a, b)$ to functions of x defined on $J := (c, d)$, in such a way that the norm $\|Tf\|_J := \left\{\int_J |Tf(x)|^2\,dx\right\}^{1/2}$ is equal to the "original" norm $\|f\|_I := \left\{\int_I |f(t)|^2\,dt\right\}^{1/2}$. That is, we want $\|Tf\|_J = \|f\|_I$ for all $f \in L^2(a, b)$.

We want $\lambda x - \mu$ to move from a to b as x moves from c to d. This gives us two equations in the two unknowns λ and μ:

$$a = \lambda c - \mu \text{ and } b = \lambda d - \mu, \text{ so } \lambda = \frac{b-a}{d-c} \text{ and } \mu = \lambda c - a = \frac{c(b-a) - a(d-c)}{d-c} = \frac{bc - ad}{d-c}.$$

If we let $t = \lambda x - \mu$, then $dt = \lambda dx$, so $dx = dt/\lambda$. Finally we look at

$$\|Tf\|^2 = \int_J |Tf(x)|^2\,dx = \int_c^d |\kappa f(\lambda x - \mu)|^2\,dx = \int_a^b |\kappa f(t)|^2\,dt/\lambda = \int_a^b |f(t)|^2\,dt$$

if $|\kappa|^2/\lambda = 1$. Thus we can choose $\kappa := \sqrt{\lambda} = \sqrt{\frac{b-a}{d-c}}$ and obtain $Tf(x) = \sqrt{\frac{b-a}{d-c}} \cdot f\left(\frac{(b-a)x - bc + ad}{d-c}\right)$.

(16.1) **Exercise:** Show that if the span of an orthonormal set $\mathcal{S}$ in a Hilbert space H is dense then $\mathcal{S}$ is a maximal orthonormal set.

Proof: Suppose not. That is, we assume that $\overline{\operatorname{span}(S)} = H$ and that there is an orthonormal set T that is larger than S. We then suppose that $y \in T$ but $y \notin S$. Then $\|y\| = 1$ and $y \perp S$, and therefore $y \perp \operatorname{span}(S)$. Since $y \in H$ and $\operatorname{span}(S)$ is dense in H, there exist vectors $s_n \in \operatorname{span}(S)$ such that $s_n \to y$. We have noticed that $y \perp \operatorname{span}(S)$. Thus $0 = \langle s_n, y\rangle \to \langle y, y\rangle = \|y\|^2 \Rightarrow y = 0$. This is a contradiction.

Question: Why is it true that *subspaces* of a Hilbert space have zero "thickness?"

Answer: "Thickness" refers here to the Questioner's picture showing a subspace drawn as a "glob" with a boundary and an interior. We can show this: if a subspace has an interior point then the subspace is all of V (if x_o is in a subset S of an inner product space V then x_o is an *interior point* of S if there exists $\delta > 0$ such that $\|y - x_o\| < \delta \Rightarrow y \in S$). So now we suppose that S is a *subspace* of V and that $x_o \in S$ is an interior point of S. Suppose that $w \in V$ and $\|w\| < \delta$. Then if we define $y := x_o + w$, then $\|y - x_o\| = \|w\| < \delta$. Therefore $y = x_o + w \in S$. Since $x_o \in S$ and S is a subspace, $w = (x_o + w) - x_o \in S$ because S is closed under addition. That is, $\|w\| < \delta \Rightarrow w \in S$. In particular, $v = 0 \in S$. Now we suppose $v \neq 0$ and $v \in V$. Then

$$\left\|\delta \frac{v}{2\|v\|}\right\| < \delta, \quad \text{so that} \quad \delta \frac{v}{2\|v\|} \in S. \quad \text{But then} \quad v = \frac{2\|v\|}{\delta}\left(\delta \frac{v}{2\|v\|}\right) \in S$$

because S is closed under scalar multiplication. Hence $S = V$.