Fourth question: Does Φ have compact support? Yes

We will use this idea to show that Φ has compact support: show that whenever $f(t)$ is in L^2 and $f(t) = 0$ a.e. in a certain interval $(a,\, b)$, then $\langle \Phi,\, f \rangle = 0$. This will show that $\Phi = 0$ a.e. *outside* $(a,\, b)$.

We will bring in some technical tools here that are due to engineers: work with "nice" functions, so we don't have to worry about the existence of various integrals, or whether inverse Fourier transforms are given by integrals or not. This will succeed whenever the "nice" functions have appropriate density properties (for us, density in L^2).

For every $g(t)$ that is infinitely differentiable, with compact support disjoint from $[M_1,\, M_2]$,

$$\int \overline{\widehat{g}(\xi)} p_N(\xi)\, d\xi = 0.$$

Everything so far has depended on knowing that $\widehat{g}$ is integrable and that p_N is bounded. Next we want to let $N \to \infty$. This time we can use Dominated Convergence. For this we take $G(\xi) := |\widehat{g}(\xi)|$. Then $|\widehat{g}(\xi) p_N(\xi)| \le G(\xi)$, and $\widehat{g}(\xi) p_N(\xi) \to \widehat{g}(\xi)\widehat{\Phi}(\xi)$. Then by the Dominated Convergence Theorem,

$$0 = \int \overline{\widehat{g}(\xi)} p_N(\xi)\, d\xi \to \int \overline{\widehat{g}(\xi)} \widehat{\Phi}(\xi)\, d\xi = \left\langle \widehat{\Phi},\, \widehat{g} \right\rangle = 2\pi \left\langle \Phi,\, g \right\rangle . \tag{$g \perp \Phi$}$$

To show that $\Phi(t) = 0$ a.e. outside $[M_1,\, M_2]$ it is enough to show that $\int_{-\infty}^{M_1} |\Phi(t)|^2\, dt + \int_{M_2}^{\infty} |\Phi(t)|^2\, dt = 0$. To do this, we define $f(t) := \Phi(t)$ if $t \notin [M_1,\, M_2]$ and define $f(t) := 0$ if $t \in [M_1,\, M_2]$.

We will do the proof by contradiction: we suppose that

$$\|f\|^2 = \int_{-\infty}^{M_1} |\Phi(t)|^2\, dt + \int_{M_2}^{\infty} |\Phi(t)|^2\, dt =: \alpha^2 > 0.$$

One of the useful features of our "nice" functions is that they are dense in L^2. If we require that they be zero when $M_1 \le t \le M_2$, they are no longer dense in all of L^2, but they *are* dense in the closed subspace of L^2 that consists of functions that are zero a.e. in the set where $M_1 \le t \le M_2$. Let us take this as a Lemma and use it.

We can thus find a "good" function g such that $\|f - g\| < \alpha/10$. About good functions g we know that $\langle \Phi,\, g \rangle = 0$. We let Φ_L denote the function $\Phi - f$. Then since $g(t)\Phi_L(t) = 0$, $\langle f,\, g \rangle = \langle \Phi,\, g \rangle = 0$. Hence

$$0 < \alpha^2 = \langle f,\, f \rangle = \langle f - g + g,\, f \rangle = \langle f - g,\, f \rangle + \langle g,\, f \rangle = \langle f - g,\, f \rangle \le \|f - g\|\, \|f\| < (\alpha/10)\alpha = \alpha^2/10.$$

This gives us the desired contradiction.

The rest of this Note is technical: the proof that *the "good" functions are dense in the subspace of* $L^2(\mathbb{R})$ *consisting of all* L^2 *functions that are zero a.e. in* $[M_1,\, M_2]$.

We will use a C^∞ function $\beta(t)$ that is zero if $|t| \ge 1/2$, positive if $|t| < 1/2$, with $\int \beta(t)\, dt = 1$. Such a function can be made by modifying the function from Appendix A2 in *From "low-pass filter" to scaling function.*

We want to show: Given $f \in L^2(\mathbb{R})$ with $f(t) = 0$ a.e. in $[M_1,\, M_2]$ and given $\alpha > 0$ find $g \in C^\infty$ such that $g(t) = 0$ in $[M_1 - \epsilon,\, M_2 + \epsilon]$ for some $\epsilon > 0$ depending on g, and $g(t) = 0$ for $|t| > R$, for some $R > 0$ depending on g, with $\|f - g\| < \alpha$. This is the technical way to say: show that we can approximate every L^2 that is zero in $[M_1,\, M_2]$ as well as we like by a "good" function.

There are several steps. First, we approximate the given $f(t)$ by two parts of itself, the part with $-T < t < M_1 - \epsilon$ and the part with $M_2 + \epsilon < t < T$, where $T > 0$ and $\epsilon > 0$ will be chosen by us, influenced by the size of the given $\alpha > 0$. We will call the sum of these parts $h(t)$. Second, we "smooth out" $h(t)$ by convolving it with $\beta(t/\delta)/\delta$, and we call that h_δ. We choose $\delta > 0$ later, but δ *must* be less than ϵ. Third, we show that h_δ is a "good" function. Fourth, we estimate the norm of the difference between h and h_δ, using two Lebesgue Facts: Minkowski's Integral Inequality and "Continuity of translation in L^p, $1 \le p < \infty$. Finally we put all the estimates together to complete the proof.

We begin by using the function $g(x)$ in Appendix A2 of *From "low-pass filter" to scaling function* to define our function $\beta(t)$:

$$\beta(t) := 2g(2t)/\int g(x)\,dx.$$

Since $g(x) = 0$ if $|x| \geq 1$, $g(2t) = 0$ if $|2t| \geq 1$, that is, if $|t| \geq 1/2$. And $\int \beta(t)\,dt = \int 2g(2t)\,dt/\int g(x)\,dx = 1$.

First Step

We know that, as $T \to \infty$, $\int_{|t|>T} |f(t)|^2\,dt \to 0$. We thus select a T so large that $\int_{|t|>T} |f(t)|^2\,dt < \alpha^2/8$. We also know that (Absolute Continuity of Integrals) there exists $\epsilon > 0$ such that

$$\int_{M_1-\epsilon}^{M_1} |f(t)|^2\,dt + \int_{M_2}^{M_2+\epsilon} |f(t)|^2\,dt < \alpha^2/8.$$

We then define

$$h(t) := \begin{cases} 0, & \text{if } -\infty < t \leq -T; \\ f(t), & \text{if } -T < t < M_1 - \epsilon; \\ 0, & \text{if } M_1 - \epsilon \leq t \leq M_2 + \epsilon; \\ f(t), & \text{if } M_2 + \epsilon < t < T; \\ 0, & \text{if } T \leq t < \infty. \end{cases}$$

"By construction" we have $\int |f(t) - h(t)|^2\,dt < \alpha^2/8 + \alpha^2/8 = \alpha^2/4$, which means $\|f - h\| < \alpha/2$.

Second Step

We next define

$$h_\delta(t) := \int \frac{1}{\delta}\beta(\frac{t-s}{\delta})h(s)\,ds \quad (=: \beta_\delta * h(t), \quad \text{if you're familiar with } \textit{convolution}).$$

Third Step

We can differentiate under the integral sign, and this leads to: $h_\delta \in C^\infty$. If $h_\delta(t) \neq 0$ there has to some s such that simultaneously $\frac{1}{\delta}\beta(\frac{t-s}{\delta}) \neq 0$ and $h(s) \neq 0$. This in turn means that $|t - s|/\delta \leq 1/2$ and that at least one of (i) $-T < s < M_1 - \epsilon$ and (ii) $M_2 + \epsilon < s < T$ is true.

In either case, $-\delta/2 \leq s - t \leq \delta/2$ so that $t - \delta/2 \leq s \leq t + \delta/2$. If (i) is true then (using half of *each* inequality) $-T < s \leq t + \delta/2$ and also $t - \delta/2 \leq s < M_1 - \epsilon$. Therefore (eliminate s) $t > -T - \delta/2 > -T - \epsilon/2$ and $t < M_1 - \epsilon + \delta/2 < M_1 - \epsilon/2$ because we required $\delta < \epsilon$. Thus in the case (i) we have $-T - \epsilon/2 < t < M_1 - \epsilon/2$.

Exercise: *Show that if* (ii) is true then $M_2 + \epsilon/2 < t < T + \epsilon/2$. Conclude that h_δ is a "good" function.

Fourth Step

Since $\int \frac{1}{\delta}\beta(\frac{t-s}{\delta})\,ds = 1$ it is true that $\int \frac{1}{\delta}\beta(\frac{t-s}{\delta})h(t)\,ds = h(t)$ and therefore

$$h(t) - h_\delta(t) = \int \frac{1}{\delta}\beta(\frac{t-s}{\delta})(h(t) - h(s))\,ds = \int \frac{1}{\delta}\beta(\frac{s}{\delta})(h(t) - h(t-s))\,ds \ \ (\text{via } \ s \mapsto t - s).$$

This means that

$$|h(t) - h_\delta(t)| \leq \int_{-\delta/2}^{\delta/2} \frac{1}{\delta}\beta(\frac{s}{\delta})|h(t) - h(t-s)|\,ds \ \text{ so } \ \|h - h_\delta\|^2 \leq \int \left(\int_{-\delta/2}^{\delta/2} \frac{1}{\delta}\beta(\frac{s}{\delta})|h(t) - h(t-s)|\,ds\right)^2 dt.$$

By Lebesgue Fact (5), **Minkowski's Integral Inequality**,

$$(*) \quad \|h - h_\delta\| \leq \left\{\int \left(\int_{-\delta/2}^{\delta/2} \frac{1}{\delta}\beta(\frac{s}{\delta})|h(t) - h(t-s)|\,ds\right)^2 dt\right\}^{1/2} \leq \int_{-\delta/2}^{\delta/2} \frac{1}{\delta}\beta(\frac{s}{\delta}) \left\{\int |h(t) - h(t-s)|^2\,dt\right\}^{1/2} ds.$$

We notice that $\left\{\int |h(t) - h(t-s)|^2 \, dt\right\}^{1/2} = \|h(t) - h(t-s)\|_{dt}$ and that $|s| \leq \delta/2$. Now we use Lebesgue Fact (10), **Continuity of norms with respect to translation**, which says that, given $\alpha > 0$, there exists $\delta_o > 0$ such that $|s| \leq \delta_o/2 \Rightarrow \|h(t) - h(t-s)\|_{dt} < \alpha/4$. Therefore by $(*)$ and by $\int \frac{1}{\delta}\beta(\frac{s}{\delta_o})\, ds = 1$, $\|h - h_{\delta_o}\| < \alpha/4$. We can replace this δ_o by $\delta := \min\{\delta_o, \epsilon/2\}$. Then $\delta < \epsilon$ and $\delta \leq \delta_o$.

Putting the estimates together

We have to show that $\|f - h_\delta\| < \alpha$. We combine our earlier estimates:

$$\|f - h_\delta\| \leq \|f - h\| + \|h - h_\delta\| < \alpha/2 + \alpha/4 < \alpha.$$

This completes the proof.