Theorem: *If $S \subseteq \mathbb{R}$ is non-empty and bounded above and $\sigma := \sup S$ does not belong to S, then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of S that is strictly increasing and such that $x_n > \sigma - 1/2^n$ for all $n \in \mathbb{Z}^+$.*

Proof: Since $\sigma - 1/2 < \sigma$, there exists $x_1 \in S$ such that $x_1 > \sigma - 1/2 = 1/2^1$. Since $\sigma \notin S$ it is also true that $\sigma > x_1$. To see how to proceed by induction we work with $y := \max\{\sigma - 1/4, x_1\}$ Since each of $\sigma - 1/4 = \sigma - 1/2^2$ and x_1 is less than σ, so is y. Thus there exists $x_2 \in S$ such that $x_2 > y$. Thus $\sigma > x_2 > x_1$ and $x_2 > \sigma - 1/2^2$.

We can now complete the proof, by induction. If $x_j \in S$, $1 \le j \le n$, have been chosen so that $x_j < x_{j+1}$ for $1 \le j < n$ and so that $x_j > \sigma - 1/2^j$ for $1 \le j \le n$ we set $y := \max\{\sigma - 1/2^{n+1}, x_n\}$. Then (as we saw before) $y < \sigma$, so there exists $x_{n+1} \in S$ such that $x_{n+1} > y$. But then $x_{n+1} > x_n$ and $x_{n+1} > \sigma - 1/2^{n+1}$. By induction we obtain the sequence we desired.

Next, a "companion" Theorem that gives a sufficient condition for knowing that $\sup S \in S$.

Theorem *Suppose that $S \subseteq \mathbb{R}$ is non-empty and bounded above and that there exists $\delta > 0$ such that whenever s_1 and s_2 are unequal elements of S, $|s_1 - s_2| \ge \delta$. Then $\sup S \in S$.*

Proof: We will use contradiction. Thus assume $\sup S \notin S$. Then $\sup S - \delta < \sup S$ and thus there exists $s_1 \in S$ such that $\sup S - \delta < s_1$. Since $supS \notin S$ and s_1 is in S, $s_1 < \sup S$. But then there exists $s_2 \in S$ such that $s_1 < s_2 < \sup S$. We can combine these inequalities: $\sup S - \delta < s_1 < s_2 < \sup S$. Then we have

$$0 < s_2 - s_1 < \sup S - s_1 < \sup S - (\sup S - \delta) = \delta, \quad \text{which contradicts our hypothesis.}$$

An application of this Theorem: Construction of the "greatest integer," or "floor" function.

Theorem: *For all $x \in \mathbb{R}$ there exists a largest integer n such that $n \le x$. This n is denoted $[x]$ and is called the greatest integer in x.*

Proof: We consider the set $S := \{m \in \mathbb{Z} : m \le x\}$, which is bounded above by x, "by construction." To show that S is non-empty we can proceed by contradiction (there is a more elegant way to see that S is non-empty).

Thus we suppose that $S = \emptyset$. This means that for all $m \in \mathbb{Z}$, $m > x$. Then $\mathbb{Z}$ is bounded below by x, and is non-empty. Thus $\tau := \inf \mathbb{Z}$ exists as a real number. Since $\tau + 1 > \tau$ there exists $M \in \mathbb{Z}$ such that $M < \tau + 1$. But then $M - 1 \in \mathbb{Z}$ and $M < \tau$, which contradicts the fact (known true by the Completeness Axiom) that τ is a lower bound for $\mathbb{Z}$. Hence S is non-empty.

Since distinct integers m and n satisfy $|m - n| \ge 1$, $\sup S \in S$, so $\sup S \in \mathbb{Z}$ and $\sup S \le x$. Since $\sup S + 1$ is not in S and $\sup S + 1 \in \mathbb{Z}$, we see that $x < \sup S + 1$. Thus among all integers that are at most x there is one that is largest, namely $\sup S$, which we call $[x]$.

An application of the greaterst integer function: **the density of the rationals** (as constructed via inductive sets) in the real numbers.

Theorem: *If x and y are real numbers and $x < y$ then there exists a rational number r such that $x < r < y$.*

Proof: By the Archimedean Property of $\mathbb{R}$ there exists a natural number n such that $n(y-x) > 1$, or $1/n < y-x$. We let $m := [nx]$. Then $m \le nx < m+1$, so that

$$\frac{m}{n} \le x < \frac{m+1}{n} =: r = \frac{m}{n} + \frac{1}{n} < \frac{m}{n} + y - x \le x + y - x = y. \quad \text{That is, } x < r < y.$$