Ask! Indicate your approach! Show your work! Good Luck! There are 5 pages, and 100 points.

(1) [10] Define *Riemann sum* in full detail. Include defining the quantities associated with defining Riemann sums!

f is defined on $[a, b]$, π is a *partition* of $[a, b]$, meaning π is a finite, strictly increasing sequence of points x_i with $a = x_0 < \cdots < x_{n_\pi} = b$, where n_π is the number of *intervals* $I_i := [x_{i-1}, x_i]$ of π, $1 \le i \le n_\pi$. We write $\Delta x_i := x_i - x_{i-1}$ for the length of I_i and $\xi \in \mathbb{R}^{n_\pi}$ is a *selection vector,* meaning that $\xi \in I_i$ for $1 \le i \le n_\pi$. Then

$$\textit{a Riemann sum for } f \textit{ on } [a, b], \text{ denoted } R(f, [a, b], \pi, \xi), \text{ is given by } R(f, [a, b], \pi, \xi) := \sum_{i=1}^{n_\pi} f(\xi_i)\Delta x_i.$$

(2) [10] State the important inequality on the difference between two Riemann sums. Define *mesh* of a partition. For arbitrary $\epsilon > 0$, with $[a, b] = [0, 1]$ and $f(x) = x$, find $\delta > 0$ such that $\text{mesh}(\pi_1) < \delta$ and $\text{mesh}(\pi_2) < \delta$ implies that the difference between any two Riemann sums over those partitions is less than 2ϵ.

$$\left|R(f, \pi_1) - R(f, \pi_2)\right| \le \sum_{i=1}^{n_{\pi_1}} \omega_{1i}\, \Delta x_i + \sum_{j=1}^{n_{\pi_2}} \omega_{2j}\, \Delta y_j \quad \text{and} \quad \text{mesh}(\pi) := \max_i \Delta x_i.$$

When $f(x) = x$ and $\pi | [a, b]$, $\omega_i = \Delta x_i \le \text{mesh}(\pi) < \delta$ that we need to choose. By the inequality,

$$\left|R(f, \pi_1) - R(f, \pi_2)\right| \le \sum_{i=1}^{n_{\pi_1}} \omega_{1i}\, \Delta x_i + \sum_{j=1}^{n_{\pi_2}} \omega_{2j}\, \Delta y_j < \delta \left(\sum_{i=1}^{n_{\pi_1}} \Delta x_i + \sum_{j=1}^{n_{\pi_2}} \Delta y_j \right) = 2\delta = 2\epsilon \text{ if we choose } \delta = \epsilon.$$

(3) [15] State the *Cauchy Criterion* for Riemann integrability. Use it (and other things you know!) to prove that if $f(x)$ is continuous on a compact interval $[a, b]$ then $f(x)$ is Riemann integrable on $[a, b]$.

A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable over $[a, b]$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all partitions π and π' of $[a, b]$, and for all selection vectors ξ and ξ' associated with π and π', respectively,

$$\text{mesh}(\pi) < \delta \text{ and } \text{mesh}(\pi') < \delta \Rightarrow |R(f, \pi, \xi) - R(f, \pi', \xi')| < \epsilon.$$

The proof part:
f is uniformly continuous so for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.
If $\text{mesh}(\pi) < \delta$ and $\text{mesh}(\pi') < \delta$, $\omega_i \le \epsilon$ and $\omega'_j \le \epsilon$, so $\left|R(f, \pi) - R(f, \pi')\right| \le \sum_{i=1}^{n_\pi} \omega_i\, \Delta x_i + \sum_{j=1}^{n_{\pi'}} \omega'_j\, \Delta y_j$ $\le \epsilon \sum_{i=1}^{n_\pi} \Delta x_i + \epsilon \sum_{j=1}^{n_{\pi'}} \Delta y_j = 2\epsilon(b - a)$ so $f(x)$ is Riemann integrable on $[a, b]$.

(4) [10] State Taylor's Theorem. Find the Taylor polynomial of degree three, with base point $x_o = 1$, for $f(x) := x^3 + 2x^2 + 3x + 4$.

Taylor's Theorem: see (5.15), page 110 in Rudin, or:
If $f(x)$ and $f^{(k)}(x)$ exist and are continuous on $[a, b]$ for $1 \le k \le n$, and if $f^{(n+1)}(x)$ exists on (a, b) then for all $x \ne x_o$, both in $[a, b]$, there exists ξ strictly between x and x_o such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_o)}{k!}(x - x_o)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_o)^{n+1}.$$

When $x = 1$, $f(x) = 10$, $f'(x) = 3x^2 + 4x + 3 = 10$, $f''(x) = 6x + 4 = 10$, $f(3)(x) = 6$, so
$\sum_{k=0}^{3} \frac{f^{(k)}(1)}{k!}(x - 1)^k = 10 + \frac{10}{1}(x - 1) + \frac{10}{2}(x - 1)^2 + \frac{6}{6}(x - 1)^3 = 10 + (x - 1) + 5(x - 1)^2 + (x - 1)^3.$

Scratch Page **Be sure to CLEARLY link work here to a problem! Put the link THERE too!**

(5) [10] **Briefly** outline the proof that, if $f(x)$ is Riemann integrable on a compact interval $[a, b]$ then $f(x)$ is bounded on $[a, b]$.

Suppose not. Then there exists x_n such that $|f(x_n)| > n$. There are infinitely many such x_n that are distinct, so there exists x^*, a limit point, and a subsequence (we still call it $\{x_n\}$) that converges to x^*. Since f is RI, we choose $\epsilon = 1$ and get $\delta > 0$ so that $\text{mesh}(\pi) < \delta \Rightarrow |R(\pi) - \int f| < 1$. If x^* is not an endpoint, we can construct a partition π, $\text{mesh}(\pi) < \delta$ with x^* in the interior of an interval I_I; if x^* is an endpoint, the argument works with I_1 or I_{n_π}. We find some x_M and some x_N in I_I such that $|f(x_M) - f(x_N)|\Delta x_I > 2$. We choose ξ_i's in the other intervals, and call ξ^M the one with $x_M^M = \xi_I$ and ξ^N the one with $x_N = \xi_I^N$. Then the non-I terms cancel, so that

$$1 > |\textstyle\int f - R(f, \pi, x_N) + R(f, \pi, x_N) - R(f, \pi, x_M)| \geq |R(f, \pi, x_N) - R(f, \pi, x_M)| - |\textstyle\int f - R(f, \pi, x_N)|,$$

and so $1 > |R(f, \pi, x_N) - R(f, \pi, x_M)| - |\int f - R(f, \pi, x_N)| > |f(x_M) - f(x_N)|\Delta x_I - 1 > 2 - 1 = 1$: contra.

(6) [10] Prove that, if $f(x)$ is defined on $[0, 1]$ by $f(0) = 0$ and by $f(x) = 1/x$ otherwise, then f is continuous almost everywhere on $[0, 1]$ but not Riemann integrable there.

Where $x \neq 0$, $1/x$ is continuous. Since $f(0) = 0$ and $f(x) \to \infty$ as $x \downarrow 0$, f is unbounded, and discontinuous at 0 alone, and $\{0\}$ is finite, hence a null set. Then (unbounded) f is not RI, and f is continuous almost everywhere in $[0, 1]$.

(7) [10] Find a Riemann sum R for $\int_1^2 (1/x)\,dx$ such that $|R - \int_1^2 (1/x)\,dx| < 1/10$.

We start with a positive number, $1/11 < 1/10$. Since $1/x$ is continuous on $[1, 2]$ it is RI there. Hence there exists a mesh size so small that Riemann sums S with a smaller mesh size satisfy $|S - \int_1^2 (1/x)\,dx| < \frac{1}{220}$. There also exists a mesh size so small that for all partitions with a smaller mesh size, $\sum_{j=1}^{n_\tau} \omega_j \Delta y_j < 1/220$. We now choose a mesh size smaller than both sizes, and a Riemann sum S on a partition τ with smaller yet mesh size. We have not yet picked our Riemann sum R. But it will have $x_i = 1 + \frac{i}{N} = \xi_i$ for some N. We thus have

$$\left|R - \int_1^2 (1/x)\,dx\right| = \left|R - S + S - \int_1^2 (1/x)\,dx\right| \leq \sum_{i=1}^{n_\pi} \omega_i\,\Delta x_i + \sum_{j=1}^{n_\tau} \omega_j\,\Delta y_j + \left|S - \int_1^2 (1/x)\,dx\right|$$

$$< \sum_{i=1}^{n_\pi} \omega_i\,\Delta x_i + \frac{1}{220} + \frac{1}{220} = \sum_{i=1}^{n_\pi} \omega_i\,\Delta x_i + \frac{1}{110}$$

$$= \sum_{i=1}^{N} \omega_i\,\Delta x_i + \frac{1}{10} - \frac{1}{11} \leq \frac{1}{10} \quad \text{if} \quad \sum_{i=1}^{N} \omega_i\,\Delta x_i \leq \frac{1}{11}.$$

Because $1/x$ decreases, $\omega_i = \frac{1}{x_{i-1}} - \frac{1}{x_i} = \frac{\Delta x_i}{x_i x_{i-1}} < \Delta x_i$. If we choose $N = 11$, $\sum_{i=1}^{N} \omega_i\,\Delta x_i = \frac{1}{11}$. Thus

$$R = \sum_{i=1}^{11} \frac{\frac{1}{11}}{1 + \frac{i}{11}} = \sum_{i=1}^{11} \frac{1}{11 + i} = \sum_{i=12}^{22} \frac{1}{i}.$$

(8) [10] Let $f(x) = 0$ if x is irrational, and let $f(x) = 1/n$ if $x = m/n$, $n > 0$ and m and n have no common factors other than ± 1. Explain in some detail why f is continuous at every irrational number x. *Prove* that f is discontinuous at every rational number. Note: $f(0) = 1$.

Proof part: If x is rational, then $f(x) \neq 0$. But every neighborhood of x contains an irrational number ξ and $f(\xi) = 0$. Thus $|f(x) - f(\xi)| = f(x)$, so taking $\epsilon = f(x)/2$ gives a contradiction.
Explain part: Let ξ be irrational, $\epsilon > 0$ given. There exists N so large that $1/N < \epsilon$. Start with $\delta_o = 1/2$. The interval $(\xi - \delta_o, \xi + \delta_o)$ contains only finitely many rational numbers m/n with (positive!) denominators less than or equal to N. These numbers make a finite set, which is closed, and ξ is not in that closed set. Hence there exists a neighborhood of ξ that contains no rational number with denominator $\leq N$. Therefore every rational number r in that neighborhood has denominator $> N$, so $f(r) < 1/N < \epsilon$. Thus f is continuous at ξ.

(9) [15] State the necessary and sufficient conditions, on a function $f(x)$ defined on a compact interval, that guarantee its Riemann integrability. The conditions should include one that involvives continuity in some manner. **Briefly** outline the proof that, if $f(x)$ *is* Riemann integrable on a compact interval then that continuity-related condition is indeed true.

If f is bounded and real-valued on a closed and bounded interval $[a, b]$, then f is Riemann integrable there if and only if f is continuous almost everywhere on $[a, b]$.

Suppose that f is RI on $[a, b]$. Let D denote the set of discontinuities of f. We let

$$\omega(x_o) := \sup\{\epsilon > 0 : \text{ for all } \delta > 0 \text{ there exists } x \text{ such that } |x - x_o| < \delta \text{ and } |f(x) - f(x_o)| \geq \epsilon.$$

The important thing is that if I is an open interval containing x_o, then $\omega_I \geq \omega(x_o)$. Thus if we have a partition of $[a, b]$, and I_i is an interval of it and $x_o \in I_i$ and $\omega(x_o) > \epsilon$, then $\omega_i > \epsilon$ except that if x_o is an endpoint of I_i, then maybe it's $\omega_{i\pm1}$ that's $> \epsilon$.

We write $D = \bigcup_n D_n$ where $D_n = \{x_o \in [a, b] : \omega(x_o) > 1/n\}$. We seek to show that each D_n is a null set. Since the union of countably many null sets is a null set this will be enough.

We will use the Oscillation Criterion. Letting $\epsilon > 0$ be given, we set $\beta = \epsilon$ and $\eta = 1/n$. We find $\delta > 0$ such that $\text{mesh}(\pi) < \delta$ implies that $\sum_{\omega_i > 1/n} \Delta x_i < \epsilon$. Then every $x_o \in D_n$ is contained in some interval I_i with $\omega_i > 1/n$, so $D_n \subseteq \cup_{\omega_i > 1/n} \Delta x_i$ and the lengths add to less than ϵ. Hence D_n is a null set.