Solutions for Test 2, Math 3283W, Spring 2002

(1) Solution:

(1) [10] State and prove the Difference of Powers Formula. Suggestion: use a literal substitution.

Difference of Powers Formula [4 pts]:
For all complex numbers a and b, and for all positive integers n,

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k.$$

Proof [6 pts]:
Let a and b be complex numbers and n a positive integer. Then
$(a-b)\sum_{k=0}^{n-1} a^{n-1-k}b^k$
$= a\sum_{k=0}^{n-1} a^{n-1-k}b^k - b\sum_{k=0}^{n-1} a^{n-1-k}b^k$
$= \sum_{k=0}^{n-1} a^{n-k}b^k - \sum_{k=0}^{n-1} a^{n-1-k}b^{k+1}$
$= \sum_{k=0}^{n-1} a^{n-k}b^k - \sum_{k'=1}^{n} a^{n-k'}b^{k'}$ (replacing $k+1$ with k')
$= a^n + \sum_{k=1}^{n-1} a^{n-k}b^k - (b^n + \sum_{k'=1}^{n-1} a^{n-k'}b^{k'})$
$= a^n - b^n$.

Grading Notes: You may replace "complex" with "real" in the above solution with no deduction. Also, for full credit to be awarded, you need to show enough work to make the cancellation in the last two lines clear.

(2) Solution:

(2) [15] Prove that, if $\{x_n\}$ is an increasing sequence of real numbers that is bounded above, then $\{x_n\}$ converges.

Suppose $\{x_n\}$ is an increasing sequence of real numbers that is bounded above. Let $S := \{y \in \mathbb{R} : y = x_n \text{ for some } n \in \mathbb{N}\}$. Then S is a nonempty set of real numbers that is bounded above. So, by the Completeness Axiom, $\sup S$ exists. Let $s := \sup S$. We will show that $x_n \to s$ [6 pts so far]. Let $\epsilon > 0$ be given. By the definition of s, we can choose $y \in S$ such that $s - \epsilon < y$. Say $y = x_N$ [4 more pts]. Then for all $n \geq N$, we have
$|x_n - s|$
$= s - x_n$ (since $x_n \leq s$ because s is an upper bound of S)
$\leq s - x_N$ (since $x_n \leq x_N$ because $\{x_n\}$ is increasing)
$< \epsilon$ (by choice of $y = x_N$).
So, $x_n \to s$.

(3) Solution:

(3) [3] What are the distinct numbers in the sequence $\{i^n\}$? [3] Find the real and imaginary parts of $z := \frac{1-i}{1+i}$. [9] Calculate $w_n := \sum_{k=0}^{n-1} z^k$ and find the distinct numbers in the sequence $\{w_n\}$.

(3 points)

$\{i^n\} = \{1, i, -1, -i\}$

(3 points)

$z = \frac{1-i}{1+i} = \frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-2i-1}{1+1+0i} = -i$

Hence $\mathrm{Re}(z) = 0$ and $\mathrm{Im}(z) = -1$.

(5 points)

Note that w_n is given by a geometric series:

$w_n = \sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z} = \frac{1-(-i)^n}{1+i}$.

(4 points)

Now since $(-i)^n$ only takes on four distinct values, we can plug those in to find the distinct values of w_n.

$$\begin{aligned}
\frac{1-1}{1+i} &= 0 \\
\frac{1+i}{1+i} &= 1 \\
\frac{1+1}{1+i} &= \frac{2}{1+i} \cdot \frac{1-i}{1-i} = \frac{2-2i}{2} = 1-i \\
\frac{1-i}{1+i} &= -i
\end{aligned}$$

(4) Solution:

(4) [10] Let $x_1 = 1$, $x_2 = \frac{1}{1+1}$, $x_3 = \frac{1}{1+\frac{1}{1+1}}$, $x_4 = \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}$, and so on. What is the limit, assuming it exists?

(4 points)

Suppose $x_n \to L$, and note that $x_{n+1} = \frac{1}{1+x_n}$.

(1 point)

Then $L = \frac{1}{1+L}$.

(3 points)

Solving for L, we have

$$\begin{aligned}
L^2 + L &= 1 \\
L^2 + L - 1 &= 0 \\
L &= \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}
\end{aligned}$$

(2 points)

Note that x_n is positive (this should be clear both from the recursion as well as the original definition). Hence we must have $L = \frac{-1+\sqrt{5}}{2}$.

(5) Solution:

(5) [10] Prove that, if $\{z_n\}$ and $\{z'_n\}$ are sequences with limits z and z' respectively, then the sequence $\{z_n z'_n\}$ has limit zz'. Give an "epsilon–N" proof, assuming that $|z_n| \leq M$ and $|z'_n| \leq M'$. You may take it for granted that $|z| \leq M$ and $|z'| \leq M'$. Suggestion: subtract and add!

Let $\epsilon > 0$ be given. [1]

$|z_n z'_n - zz'| = |z_n z'_n - z_n z' + z_n z' - zz'| \leq |z_n z'_n - z_n z'| + |z_n z' - zz'| = |z_n||z'_n - z'| + |z_n - z||z'|$.

Then $|z_n z'_n - zz'| \leq M|z'_n - z'| + |z_n - z|M'$. [4]

Also, $|z'_n - z'| < \epsilon/(M+1)$ if $n \geq N_1$ and $|z_n - z| < \epsilon/(M'+1)$ if $n \geq N_2$. [3]
Then if $n \geq N := \max\{N_1, N_2\}$,

$$|z_n z'_n - zz'| \leq M|z'_n - z'| + |z_n - z|M' < M\epsilon/(M+1) + \epsilon M'/(M'+1) < \epsilon. \quad [2]$$

(6) Solution:

(6) [20] Prove that, for all positive integers n, if $0 \leq r \leq n$ then $\rho_{nr} := \left(\frac{n+1}{n}\right)^r \frac{n+1-r}{n+1} \leq 1$. Suggestion: calculate ρ_{n0}, then calculate $\rho_{nr} - \rho_{n,r+1}$.

$\rho_{n0} := \left(\frac{n+1}{n}\right)^0 \frac{n+1-0}{n+1} = 1 \leq 1$. [4]

$$\begin{aligned}
\rho_{nr} - \rho_{n,r+1} &= \left(\frac{n+1}{n}\right)^r \frac{n+1-r}{n+1} - \left(\frac{n+1}{n}\right)^{r+1} \frac{n+1-r-1}{n+1} \\
&= \left(\frac{n+1}{n}\right)^r \frac{1}{n+1} \left[n+1-r - \left(\frac{n+1}{n}\right)(n+1-r-1)\right] \\
&= \left(\frac{n+1}{n}\right)^r \frac{1}{n+1} \left[n+1-r - \left(1+\frac{1}{n}\right)(n-r)\right] \\
&= \left(\frac{n+1}{n}\right)^r \frac{1}{n+1} \left[n+1-r - (n-r) - \frac{n-r}{n}\right] \\
&= \left(\frac{n+1}{n}\right)^r \frac{1}{n+1} \left[1 - 1 + \frac{r}{n}\right] \\
&= \left(\frac{n+1}{n}\right)^r \frac{r}{n(n+1)}. \quad [9]
\end{aligned}$$

Since $\left(\frac{n+1}{n}\right)^r \frac{r}{n(n+1)} \geq 0$, (the inequality is $>$ if $r > 0$),

ρ_{nr} decreases as r increases. Thus $\rho_{nr} \leq \rho_{n0} = 1$. [7]

(7) Solution:

(7) [10] Suppose that A is a non-empty subset of $\mathbb{R}$, that $B \subseteq \mathbb{R}$ is bounded above, and suppose that $A \subseteq B$. Prove that sup A and sup B both exist, and that sup $A \leq$ sup B.

(2 points) We are given that $A \neq \emptyset$.

Also, suppose M is an upper bound for B. Then $a \in A \Rightarrow a \in B \Rightarrow a \leq M$, so A is bounded above.

Thus sup A exists by the Completeness Axiom.

(2 points) We are given that B is bounded above.

Also, since $A \neq \emptyset$, there is some element a in A. But then a is also in B, since $A \subseteq B$. Hence $B \neq \emptyset$.

Thus sup B also exists (again, by the Completeness Axiom).

(6 points) Suppose (in order to get a contradiction) that sup $A >$ sup B. Then there is some element $a \in A$ with sup $B \leq a$. But $a \in A \Rightarrow a \in B$, so $a \leq$ sup B, which is a contradiction. So we must have sup $A \leq$ sup B.

(8) Solution:

(8) [10] Let $x_1 = \frac{1}{10}$ and $x_{n+1} = 1 - x_n$ for $n \geq 1$. Prove that $\{x_n\}$ has no limit as $n \to \infty$.

(2 points)

Note that $x_{n+2} = 1 - x_{n+1} = 1 - (1 - x_n) = x_n$. Hence

$$x_n = \begin{cases} \frac{9}{10} & \text{if } n \text{ is even;} \\ \frac{1}{10} & \text{if } n \text{ is odd.} \end{cases}$$

(2 points)

Suppose that x_n does converge to some limit L, in order to get a contradiction.

(1 point)

Then there is some $M \in \mathbb{N}$ so that $|x_n - L| < \frac{2}{5}$ whenever $n \geq M$ (any $\epsilon \leq \frac{2}{5}$ will do).

(4 points)

Since n can be even or odd for arbitrarily large values of n, both of the following must hold for any $n \geq M$:

For n even	For n odd
$\lvert\frac{9}{10} - L\rvert < \frac{2}{5}$	$\lvert\frac{1}{10} - L\rvert < \frac{2}{5}$
$-\frac{2}{5} < \frac{9}{10} - L < \frac{2}{5}$	$-\frac{2}{5} < \frac{1}{10} - L < \frac{2}{5}$
$-\frac{13}{10} < -L < -\frac{1}{2}$	$-\frac{1}{2} < -L < \frac{1}{10}$
$\frac{1}{2} < L < \frac{13}{10}$	$-\frac{1}{10} < L < \frac{1}{2}$

(1 point)

But then $\frac{1}{2} < L < \frac{1}{2}$, which is a contradiction. Thus x_n must not converge.

(9) Solution:

(9) [10] State the n-th root Theorem. Show that if $x > 1$ then $x^{1/n} \to 1$ as $n \to \infty$. You may use without proof that $x^{1/n} > 1$ if $x > 1$.

n-th Root Theorem [4 pts]:
For all positive integers n and all non-negative real numbers y, there exists exactly one non-negative real number x such that $x^n = y$.

Suppose $x > 1$. Let $\epsilon > 0$ be given. Choose a positive integer N such that $N > (x-1)/\epsilon$. Suppose $n \geq N$. Then $n > (x-1)/\epsilon$ and so $1 + n\epsilon > x$. Now, by the Binomial Theorem, $(1+\epsilon)^n > 1 + n\epsilon$ (the first two terms in the expansion). So, $(1+\epsilon)^n > x$. It follows from the Difference of Powers Formula that $1 + \epsilon > x^{1/n}$, and so $|x^{1/n} - 1| = x^{1/n} - 1 < \epsilon$. We've proved that $|x^{1/n} - 1| < \epsilon$ for all $n \geq N$. This means that $x^{1/n} \to 1$.

(10) Solution:

(10) [20] Prove that, if S is a bounded non-empty set of real numbers and $\sup S \notin S$, then there exists a sequence $\{s_n\}$ such that $s_n \in S$ for all natural numbers n, and $\lim_{n\to\infty} s_n = \sup S$. Hint: Consider the numbers $\sup S - (1/n)$.

Suppose that S is a bounded non-empty set of real numbers such that $\sup S \notin S$. For each positive integer n, we can choose an element $s_n \in S$ such that $s_n > \sup S - \frac{1}{n}$. We can do this by the definition of $\sup S$ since $\sup S - \frac{1}{n} < \sup S$ [10 pts so far]. We have defined our sequence $\{s_n\}$, now we must show that $s_n \to \sup S$. Let $\epsilon > 0$ be given. Choose a natural number N such that $N > 1/\epsilon$. Then, for all $n \geq N$, we have
$|s_n - \sup S|$
$= \sup S - s_n$ (since $s_n < \sup S$)
$< \frac{1}{n}$ (by our choice of s_n)
$< \epsilon$ (since $n > 1/\epsilon$).
We have proven that $s_n \to \sup S$.