Ask! Indicate your approach! Show your work! Good Luck! There are 5 pages, and 100 points.

(1) [10] Define *curve in* $\mathbb{R}^k$ and *length* of a curve in $\mathbb{R}^k$. Prove that the graph of a function of bounded variation on $[a, b]$ is the image of a curve in $\mathbb{R}^2$ that has *finite* length.

A *curve in* $\mathbb{R}^k$ is a continuous function $\varphi : [a, b] \to \mathbb{R}^k$, where $[a, b]$ is a compact interval in $\mathbb{R}$. The *length* of a curve φ in $\mathbb{R}^k$ is denoted $L(\varphi)$ and given by

$$L(\varphi) := \sup_{\pi|[a,b]} \sum_{j=1}^{n_\pi} |\varphi(x_k) - \varphi(x_{k-1})|, \quad \text{sup of the lengths of certain inscribed polygonal paths.}$$

Given that f has bounded variation on $[a, b]$ let $\varphi(x) := (x, f(x))$, for $a \le x \le b$. Then for any $\pi|[a, b]$ $|\varphi(x_k) - \varphi(x_{k-1})| \le |x_k - x_{k-1}| + |f(x_k) - f(x_{k-1})|$ so each sum competing to be $L(\varphi)$ is at most

$$\sum_{k=1}^{n_\pi} (|x_k - x_{k-1}| + |f(x_k) - f(x_{k-1})|) \le (b - a) + V, \quad \text{where } V = \text{var}\,(f, [a, b]), \quad \text{so } L(\varphi) < \infty.$$

People who earned points on this problem got them, but the Test 3 "possible" was reduced by its 10 points.

(2) [10] Define *equicontinuous.* Suppose that $\{f_n\}$ is a uniformly convergent sequence of continuous functions on a compact set K. Prove that $\{f_n\}$ is equicontinuous on K.

A family $\mathcal{F}$ of functions defined on a metric space X is *equicontinuous* if for all $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$. The absolute value here *could* be replaced by $d_Y(f(x), f(y))$ if the values of the function in $\mathcal{F}$ are in a metric space Y.

The functions f_n converge to a continuous function f, which is uniformly continuous. Thus to every $\epsilon > 0$ there exists $\delta_\omega > 0$ such that $d(x, y) < \delta_\omega \Rightarrow |f(x) - f(y)| < \epsilon/3$. There is also n_ϵ so large that $n \ge N_\epsilon \Rightarrow |f_n(x) - f(x)| < \epsilon/3$ for all $x \in K$. Thus if $n \ge N_\epsilon$, and $d(x, y) < \delta_\omega$,

$$|f_n(x) - f_n(y)| \le |f_n(x) - f(x)| + |f(x) - f(y)| + |f(y) - f_n(y)| < \epsilon.$$

For $n < N_\epsilon$, there exists $\delta_n > 0$ such that $d(x, y) < \delta_n \Rightarrow |f_n(x) - f_n(y)| < \epsilon$.
If we now define $\delta := \min\{\delta_1, \ldots, \delta_{N_\epsilon - 1}, \delta_\omega\}$, $d(x, y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon$ for all n. Thus $\{f_n\}$ is equicontinuous.

(3) [15] State the two *Stone-Weierstrass Theorems.* Prove that the collection of all *trigonometric polynomials,* namely the functions of the form $P(x) = \sum_{n=-N}^{N} c_n e^{inx}$, where N can be any non-negative integer and the c_n are complex, is dense in the complex-valued version of $C[0, 1]$.

The Theorems in question are (7.32) and (7.32). The other part is an application of (7.33). The family $\mathcal{P}$ here is closed under conjugation, consists of continuous functions and does not vanish at any point because $\mathcal{P}$ contains the constant functions. If $x \ne y$ and both points are in $[0, 1]$ then $P(t) = e^{it}$ is in $\mathcal{P}$ and $e^{ix} \ne e^{iy}$ (proof of this *not* required). Thus $\mathcal{P}$ satisfies the hypotheses of (7.33), and so is dense in the complex-valued version of $C[0, 1]$.

(4) [10] State the *Weierstrass Theorem* and **describe in some but not too much detail** how it is proved.

The Theorem is (7.26). It is proved by first reducing to the case when $f(0) = 0 = f(1)$, by subtracting $f(0)+x(f(1)-f(0))$. Calling such functions $g(x)$ we formed the convolution of g with the polynomials $Q_n(x) := c_n(1 - x^2)^n$ where c_n was chosen to make $\int_{-1}^{1} Q_n = 1$. We found that $c_n < n+1$ and found that if $0 < \delta \le 1$, then $Q_n(t) \to 0$ uniformly as $n \to \infty$. we showed by calculation that $P_n(x) = \int_0^1 Q_n(x - t)g(t)\,dt$ is a polynomial for each n. We could then write

$$P_n(x) - g(x) = \int_0^1 Q_n(x - t)g(t)\,dt - \int_{-1}^{1} g(x)Q_n(t)\,dt = \int_{-1}^{1} (g(x - t) - g(x))Q_n(t)\,dt$$

(since $[-1, 1]$ contains the interval $[x - 1, x]$ for every $x \in [0, 1]$). Near $t = 0$ (i.e. for $|t| \le \delta$) we use the fact that by uniform continuity $|g(x - t) - g(x)|$ is small, and away from 0 by a distance $\delta > 0$ we use the fact that $Q_n(t)$ is small. Thus the integral is small for n large enough.

Scratch Page **Be sure to CLEARLY link work here to a problem! Put the link THERE too!**

(5) [10] Prove that an equicontinuous sequence that converges pointwise on a compact set converges uniformly.

Let $\epsilon > 0$ be given. If $\{f_n\}$ is equicontinuous on a compact set K we have to show there exists N such that $m \geq N$ and $n \geq N$ together imply $|f_n(x) - f_m(x)| < \epsilon$, First we find $\delta > 0$ such that $(*)$ $d(x, y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon$ for all n. From the open cover formed by all balls $B_{\delta/2}(x)$, $x \in K$, we select a finite subcover $B_{\delta/2}(x_k)$, $1 \leq k \leq K$. By finiteness, $\{f_n\}$ converges uniformly on $\{x_1, \ldots, x_K\}$. Hence $(**)$ there exists N such that $n \geq N$ and $m \geq N \Rightarrow |f_n(x_k) - f_m(x_k)| < \epsilon$ for all k, $1 \leq k \leq K$. Each x is in some $B_{\delta/2}(x_k)$. For that k,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(x)| < 3\epsilon, \quad \text{by } (*) \text{ and } (**).$$

Done.

(6) [10] For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$ converge? On what intervals does it converge uniformly? *Why?*

The series converges at every $x \in \mathbb{R}$. Convergence at zero is immediate since every term is zero. If $x \neq 0$, $\frac{|x|}{1+n^2x^2} \leq \frac{1}{n^2|x|}$ so convergence follows by Comparison and the p-test, $p = 2$. If $|x| \geq \delta > 0$, $\frac{|x|}{1+n^2x^2} \leq \frac{1}{n^2\delta} =: M_n(\delta)$ so the series converges uniformly on $[\delta, \infty)$ and on $(-\infty, \delta]$ by the Weierstrass M-Test. The series does not converge uniformly on any interval of positive length that contains zero in its closure. The largest absolute value of $\frac{x}{1+n^2x^2}$ occurs when $x = \pm 1/n$ (the derivative is $\frac{1-n^2x^2}{(1+n^2x^2)^2}$) and the absolute value there of that term is $1/2n$. If our interval contains $[0, 1/N]$, it contains $1/M$ for all large M. To have *uniform* convergence we would have to have, for any $\epsilon > 0$, that there exists M_ϵ such that for all $M \geq M_\epsilon$, $\sum_{n=M}^{\infty} \frac{x}{1+n^2x} < \epsilon$ for all $x \in [0, 1/N]$. We may assume $M_\epsilon > N$. Let us choose $\epsilon = 1/10$ and $x = \frac{1}{2M_\epsilon}$. Then

$$\sum_{n=M}^{\infty} \frac{1/2M_\epsilon}{1+n^2/4M_\epsilon^2} > \sum_{n=M_\epsilon}^{2M_\epsilon} \frac{1/2M_\epsilon}{1+n^2/4M_\epsilon^2} = \sum_{n=M_\epsilon}^{2M_\epsilon} \frac{2M_\epsilon}{4M_\epsilon^2+n^2} > M_\epsilon \frac{2M_\epsilon}{4M_\epsilon^2+4M_\epsilon^2} = \frac{1}{4} > \frac{1}{10}.$$

(7) [10] State and prove the *Weierstrass M-Test.*

The Theorem is (7.10). To prove it we want to show that $\{F_n(x)\}$ is uniformly Cauchy, where $F_n(x) := \sum_{k=1}^{n} f_k(x)$, and we are given that $|f_k(x)| \leq M_k$ for all x (of interest) and that $\sum M_k < \infty$.

Let $\epsilon > 0$ be given. Then there exists N_ϵ such that $n \geq N_\epsilon \Rightarrow \sum_{k=n}^{\infty} M_k < \epsilon$. Then for $m \geq N_\epsilon$ and $n \geq N_\epsilon$, (assuming $n < m$) $|F_m(x) - F_n(x)| \leq \sum_{k=n+1}^{m} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_k < \epsilon$.

(8) [15] For which R do the following series converge (1) absolutely (2) uniformly, for all z with $|z| < R$? Why?

(a) $\displaystyle\sum_{n=0}^{\infty} 2^n z^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{2^{n^2}}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{z^{n^2}}{2^n}$.

(a) (1): $R = 1/2$ (radius of convergence can be seen "by inspection").
(2): $R \leq \rho < 1/2$. M-Test. Geom. series converge uniformly on *compact* subsets of the interions of their discs of convergence.

(b) (1): $R = \infty$: $\frac{1}{R} = \limsup \sqrt[n]{1/2^{n^2}} = \limsup 1/2^n = 0$.
(2): $R \leq \rho < \infty$. M-Test. *No* power series (except the zero one) can converge unufirmly on the whole complex plane!

(c) (1): $R = 1$: $\frac{1}{R} = \limsup \sqrt[n^2]{1/2^n} = \limsup 1/2^{1/n} = 1$.
(2): $R \leq 1$. M-Test. Converges uniformly on the closure of its disc of convergence!

(9) [10] Prove that if a power series $\sum_{n=0}^{\infty} c_n z^n$ converges for some $z_o \neq 0$ then $\sum_{n=0}^{\infty} c_n z^n$ converges absolutely for all z with $|z| < |z_o|$.

If $\sum_{n=0}^{\infty} c_n z_o^n$ converges and $z_o \neq 0$ then $c_n z_o^n \to 0$, so $|c_n z_o^n| \leq M$ for some M and all n. Suppose $|z| < |z_o|$. Then $|z/z_o| < \frac{|z|+|z_o|}{2|z_o|} =: \rho < 1$. And $|c_n z^n| = |(\frac{z}{z_o})^n c_n z_o^n| \leq M|\frac{z}{z_o}|^n < M\rho^n$. Therefore by comparison with a Geometric Series, $\sum_{n=0}^{\infty} c_n z^n$ converges absolutely. This holds for each z with $|z| < |z_o|$, as desired.