(1) [10] State the Schwarz Inequality for vectors in $\mathbb{R}^k$ and show that it is true when both vectors have length 1.

(2) [10] Give a *self-contained* proof that there is no rational number whose square is 5.

(3) [10] Define *open cover* of a set $E \subseteq \mathbb{R}^k$. Show that if $\mathcal{C}$ is an open cover of E then there exists an open cover $\mathcal{B}$ of E with these properties: for each $x \in E$ there exists exactly one $r > 0$ such that $B_r(x) \in \mathcal{B}$, and there exists $C \in \mathcal{C}$ such that $B_r(x) \subseteq G$. That is, $\mathcal{B}$ consists of balls, exactly one centered at each $x \in E$, each ball being contained in an open set from $\mathcal{C}$.

(4) [15] Define *uncountable.* Prove that the set of all subsets of $\mathbb{N}$ is an uncountable set.

(5) [10] Define *metric* on a set X. Show that $d(x, y) := \max\{\,|x_1 - y_1|, |x_2 - y_2|\,\}$ is a metric on $\mathbb{R}^2$.

(6) [10] Prove that if $x > 0$ and $y > 0$ and n is a positive integer then $x^n < y^n$ if and only if $x < y$.

(7) [10] Define *compact set* in a topological space. Show that every finite set is compact.

(8) [15] Define *limit point* and *isolated point* of (or *for*) a set E in a metric space X. Use Rudin's definitions. **Briefly** explain why, if x_o is a limit point of E, then every neighborhood of x_o contains infinitely many points of $E \setminus \{x_o\}$. How is the definition of "isolated point" related to that of "limit point?"

(9) [10] Using Rudin's definitions, prove that the complement of an open set is a closed set.

(1) [10] Prove that every compact subset K of a metric space X is a closed set.

(2) [10] Define *perfect set.* State the Theorem about the size of perfect sets in $\mathbb{R}^k$ and **very briefly** describe how the Theorem is proved.

(3) [15] Prove, *using the definition*, that $[0, 1] \subseteq \mathbb{R}$ is compact.

(4) [10] Define *convergent sequence,* in a metric space. Prove that $\{1/2^n\}$ converges to 0.

(5) [10] Prove that $\{(-1)^n\}$ diverges.

(6) [10] Define *convergent series of complex numbers.* Prove that if $\sum a_n$ converges then $a_n \to 0$.

(7) [10] What are the partial sums of the geometric series? For which complex z does the geometric series converge? Why?

(8) [15] Define *monotone sequence.* Suppose that $\{s_n\}$ satisfies (1) $s_1 = 3$ and (2) $s_{n+1} = \sqrt{1 + s_n}$. Prove that $\{s_n\}$ is monotone and prove that $\{s_n\}$ converges.

(9) [10] Define *Cauchy sequence.* Prove that a convergent sequence (in a metric space) is a Cauchy sequence.

(1) [10] Prove that the image of a compact metric space X under a continuous $f : X \to Y$ is a compact set, if Y is a metric space.

(2) [15] Define *absolutely convergent series.* Prove that if a series converges absolutely, then it converges..

(3) [10] State and prove the Cauchy Condensation Test.

(4) [10] Let $\{x_n\}$ be a sequence of real numbers. Define *the limit superior,* $\limsup_{n\to\infty} x_n$, of $\{x_n\}$. Prove that, if $A > \limsup_{n\to\infty} x_n$ then there exists N so large that $n \geq N \Rightarrow x_n < A$.

(5) [10] Prove *directly* that $\sum_n (-1)^n a_n$ converges if $a_n \downarrow 0$. Examine the even and the odd partial sums.

(6) [10] Define *differentiable function* (at a point $x_o \in \mathbb{R}$). Assume the function is complex-valued. Prove that if f is differentiable at x_o then f is continuous at x_o.

(7) [10] State and prove the Cauchy Mean Value Theorem. State any "named" theorem that you use.

(8) [10] Define (generally) *continuous function* (at a point x_o).

Prove or disprove: $f(x, y) := \begin{cases} \frac{xy^2}{x^2+y^2}, & \text{if} \quad x^2 + y^2 > 0; \\ 0, & \text{if} \quad x^2 + y^2 = 0 \end{cases}$ is continuous at $(0, 0)$.

(9) [15] Test the following series for convergence.
Important: If the series has x, test for all positive x. Identify the convergence tests you use!!

$$(a)\ \sum_1^\infty \frac{x^n (n!)^2}{(2n)!}; \quad (b)\ \sum_1^\infty \frac{n^2}{2^n}; \quad (c)\ \sum_1^\infty \frac{x^n}{\sqrt{n}}; \quad (d)\ \sum_1^\infty \frac{(-1)^n x^n}{n}; \quad (e)\ \sum_1^\infty \frac{1}{(1+\frac{1}{n})^n}.$$