(1) [10] State and prove the Theorem relating positive variation, negative variation and (total) variation.

(2) [12] Define *outer measure* on $\mathbb{R}^n$ and show that if $|E|_e < \infty$ and $\epsilon > 0$ is given, then there exists an open set $G \supseteq E$ such that $|G|_e < |E|_e + \epsilon$.

(3) [10] Define *measurable set* in $\mathbb{R}^n$, and show that intervals are measurable and that the measure of an interval is equal to its volume. A *description* of the proof (covering all main points!) is preferred.

(4) [10] Given $f : \mathbb{R} \to \mathbb{R}$ that is continuous, prove that the set $E := \{(x, y) \in \mathbb{R}^2 : f(x) = y\}$ is Lebesgue measurable. Find $|E|$. Suggestion: work with $E \cap B_k(0)$ first.

(5) [10] Define *sigma-algebra.* Given a set X, what are the largest and smallest σ-algebras of subsets of X? Does $\mathbb{Z}$ have any other σ-algebras?

(6) [10] Define *norm* on a vector space (assume the scalars are real). Prove that a *linear* map $T : V \to W$ between two normed spaces is continuous if and only if T is continuous at 0.

(7) [10] Prove that if G is an open set in $\mathbb{R}^n$ then G can be written as the union of countably many nonoverlapping cubes. State the version that holds in $\mathbb{R}$ but not in $\mathbb{R}^n$ with $n > 1$.

(8) [10] State and prove a characterization of Lebesgue measurability that tells us how to obtain the Lebesgue measurable sets from the Borel measurable sets.

(9) [10] Let f and α be functions defined on $[a, b]$. Define "$\int_a^b f(x)\, d\alpha(x)$." Show that if $\int_a^b f(x)\, d\alpha(x)$ exists, then $\int_a^b \alpha(x)\, df(x)$ exists as well. A *description* of the proof (covering all main points!) is preferred.

(10) [10] Prove that the measures of an decreasing sequence of Lebesgue measurable sets decrease to the Lebesgue measure of their intersection, provided an extra hypothesis holds. In addition, show that the extra hypothesis cannot be dropped.

(1) [15] Carefully define the *Lebesgue integral of a non-negative function.* State the Theorem about the existence of the Lebesgue integral of a non-negative function and **briefly** outline its proof, covering all main points.

(2) [10] State Carathéodory's Theorem and prove half of it.

(3) [10] State the Theorem about Lipschitz maps and measurability and give a *description* of the proof (covering all main points!).

(4) [10] Suppose $\mathcal{F}$ is a family of lower semicontinuous real-valued functions f. Prove that $\sup\limits_{f \in \mathcal{F}} f$ is a lower semicontinuous extended-real-valued function.

(5) [10] State and prove the Monotone Convergence Theorem for non-negative measurable functions.

(6) [10] Prove that if E_1 and E_2 are measurable subsets of $\mathbb{R}$ then $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$.

(7) [10] State the Theorems of Egorov and Lusin. For what purpose is the first used in the proof of the second?

(8) [10] Define *convergence in measure.* Show that if $\{f_k\}$ converges in measure to f, then $\{f_k\}$ is "Cauchy in measure."

(9) [10] State and prove Fatou's Lemma.

(10) [10] Prove that $Lf := f(1/4) - 2f(1/2) + f(3/4)$ is a continuous linear functional on $C[0, 1]$ and guess (correctly) $\alpha(x)$ such that $Lf = \int_0^1 f(x) d\alpha(x)$ for all $f \in C[0, 1]$.