Theorem 3.7 asserts that the set of subsequential limits of a sequence in a metric space is a closed set.

Proof: We let E^* denote the set of subsequential limits of a sequence $\{x_n\}$ in a metric space (X, d). If E^* is empty [e.g., $\mathbb{N}$ in $(\mathbb{R}, |x-y|)$] or has no limit points then E^* is closed. We now assume that E^* has a limit point, that we'll call q. We will show that $q \in E^*$.

- This part is actually the proof that whenever q is a limit point for a set S then there exists in S a sequence of distinct points that converges to q. In it we can replace E^* by any subset $S \subseteq X$ with $q \in S'$. A review of the Axiom of Choice will come after we finish the proof of **3.7**.

For each $r > 0$, we define $E^*(r) := \{p \in X : p \in E^* \text{ and } 0 < d(p, q) < r\}$. Since q is a limit point for E^*, each $E^*(r)$ is non-empty. By the Axiom of Choice there is a "choice function" $c : \mathbb{R}^+ \to X$ such that for each $r > 0$, $c(r) \in E^*(r)$. Now, to use the Recursion Theorem we define a function $H : \mathbb{R}^+ \to \mathbb{R}^+$ by $H(r) := \frac{1}{2}d(c(r), q)$. This may need some explanation. Starting with $r > 0$ we use the choice function c to obtain the element $p = c(r) \in E^*(r)$. Then we put $H(r)$ equal to $\frac{1}{2}d(p, q)$. This gives us a point $y = c(H(r)) \in E^*(H(r))$. Since $y \in E^*(H(r))$, $0 < d(y, q) < H(r) = \frac{1}{2}d(p, q)$. The function H "does the induction step."

We may now apply the Recursion Theorem [with $H(r, k) = H(r)$]. We choose $r = 1$. Then there exists a unique sequence r_k such that $r_0 = 1$ and $r_{k+1} = H(r_k)$ for all $n \in \mathbb{N}$. We then obtain a sequence $\{p_k\}$ in E^* by putting $p_k := c(r_k)$. Then $d(p_k, q) < r_k$.

$$\text{(1)} \qquad \text{Since } r_{k+1} = H(r_k) = \frac{1}{2}d(c(r_k), q) = \frac{1}{2}d(p_k, q) \text{ we have } 0 < d(p_{k+1}, q) < \frac{1}{2}d(p_k, q).$$

By using the triangle inequality we can show that $d(p_{k+1}, p_k) > \frac{1}{2}d(p_k, q) > 0$. Then by induction we can show that $d(p_k, p_m) > 0$ if $m \neq k$. Thus the points p_k are distinct.

Let us show that if $m > 0$ then $d(p_m, q) < \frac{1}{2^m}d(p_0, q)$. We have just shown this is true for $m = 1$ by letting $k = 0$ in (1). Given that $d(p_m, q) < \frac{1}{2^m}d(p_0, q)$, we take $k = m$ in (1) and find that

$$d(p_{m+1}, q) < \frac{1}{2}d(p_m, q) < \frac{1}{2} \cdot \frac{1}{2^m}d(p_0, q) = \frac{1}{2^{m+1}}d(p_0, q).$$

It follows that $p_k \to q$. •

It remains to show that we can find a strictly increasing sequence $\{n_k\}$ of indices such that $x_{n_k} \to q$. We will do this by choosing a suitable n_k that satisfies $d(x_{n_k}, p_k) < \frac{1}{2}d(p_k, q)$. If we can succeed in doing this we will have $d(x_{n_k}, q) \le d(x_{n_k}, p_k) + d(p_k, q) < \frac{3}{2}d(p_k, q) < \frac{2}{2^k}d(p_0, q) \to 0$.

For each $k \in \mathbb{N}$ there is a strictly increasing sequence $n_{k,j}$ of indices such that $x_{n_{k,j}} \to p_k$. Thus for each k the set of such subsequences is non-empty. By using the Axiom of Choice we obtain a"choice function" that gives us, for each $k \in \mathbb{N}$, a subsequence $x_{n_{k,j}} \to p_k$. We can now define a *set* $S_k \subseteq \mathbb{N}$ given by

$$S_k := \{n_{k,j} : j \in \mathbb{N}\} \text{ because the } n_{k,j} \text{ increase } \textit{strictly} \text{ with } j.$$

We want, for each $k \in \mathbb{N}$, to find $n_k \in S_k$ such that $n_k < n_{k+1}$. These n_k will need to be chosen so they are "suitable" as defined in a previous paragraph. We can use the Recursion Theorem once we find an appropriate function $H : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. This one should work: we use the Well-Ordering Theorem to choose (when $k \ge 1$)

$$\text{(2)} \qquad H(m.k-1) \text{ to be the first } n_{k,j} \in S_k \text{ such that } n_{k,j} > m \text{ and } d(x_{n_{k,j}}, p_k) < \frac{1}{2}d(p_k, q).$$

That is, $H(m.k-1) = n_{k,j_m}$ where j_m is the *first* j such that $n_{k,j}$ satisfies the conditions in (2). This really is a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ because $k-1$ runs through $\mathbb{N}$ as k runs through the positive integers.

We have to check that this does work. We apply the Recursion Theorem and obtain a unique sequence $\{n_k\}$ such that $n_0 = n_{0,0}$ and such that $n_{k+1} = H(n_k, k)$. Then n_0 has been found, and for $k \ge 1$ we have, with $m := n_{k-1}$,

$$n_k = H(n_{k-1}, k-1) = H(m.k-1) = n_{k,j_m} \in S_k \text{ and } n_{k-1} = m < n_{k,j_m} = n_k.$$

This means that $\{x_{n_k}\}$ is a subsequence. Moreover, since $n_k = H(n_{k-1}, k-1) = n_{k,j_m}$, as in (2), $d(x_{n_k}, p_k) < \frac{1}{2}d(p_k, q)$. As we noted before, this means that $d(x_{n_k}, q) < \frac{2}{2^k}d(p_0, q) \to 0$ as $k \to \infty$. Thus $q \in E^*$.

Remarks The Axiom of Choice can be stated many ways. In this note we used it in the "indexed family" fashion. That is, given sets X and S and a function that assigns to each $s \in S$ a *non-empty* set $E_s \subseteq X$, There exists a function (choice function) $c : S \to X$ such that for all $s \in S$, $c(s) \in E_s$.

Our first use of the Axiom of Choice used the choice function c explicitly. Our second use of the Axiom of Choice used the choice function c implicitly. Thus we could have stated the Axiom of Choice this way: If, for each $s \in S$, E_s is a non-empty subset of X, then there exists a subset $Y \subseteq X$ that contains an element of each set E_s. We use phrases such as: "We choose an element $x_s \in E_s$ for each s," instead of referring to $x_s := c(s)$.

Previously, we have used a "self-indexed" version of the Axiom of Choice: If $\mathcal{C}$ is a family of non-empty subsets of a set X then there exists a choice-function $c : \mathcal{C} \to X$ such that for all $A \in \mathcal{C}$, $c(A) \in A$. This can be stated implicitly as: If $\mathcal{C}$ is a family of non-empty subsets of a set X then there exists a set $Y \subseteq X$ that contains an element of each set A in $\mathcal{C}$.

There is a "more general" version of the Axiom of Choice. If S is a set and for each $s \in S$ there is a non-empty set X_s associated with s then there exists a set Y that contains exactly one element of each set X_s. This seems a bit vague, for there is no "home" for Y. Actually there is, though. We define the *Cartesian product* of the sets X_s as the set of *all* functions $f : S \to \bigcup_{s \in S} X_s$ that have the property that $f(s) \in X_s$ for all $s \in S$. The Axiom of Choice is then the *assumption* that the Cartesian product exists! The notation used is $\prod_{s \in S} X_s$. Two elements σ and τ of $\prod_{s \in S} X_s$ are *equal* if they are "equal coordinatewise:" for every $s \in S$, $\sigma_s = \tau_s$. In other words, the function f in the definition of $\prod_{s \in S} X_s$ is written in "sequential" form.

The implicit versions are easier to use unless there is the confusing possibility that the same element of X might be chosen for *different* sets. The use of "choice functions" explicitly avoids that possible confusion.