For intervals $I \subseteq \mathbb{R}^n$ and $J \subseteq \mathbb{R}^m$ we know $|I \times J| = |I||J|$.
Lemma: *For any sets* $A \subseteq \mathbb{R}^n$ *and* $B \subseteq \mathbb{R}^m$, $|A \times B|_e \le |A|_e|B|_e$.
Proof: Say $A \subseteq \bigcup_p I_p$ and $B \subseteq \bigcup_q J_q$, coverings by intervals in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Then $A \times B \subseteq \bigcup_{p,q} I_p \times J_q$, a covering by intervals in $\mathbb{R}^{n+m}$. Thus $|A \times B|_e \le \sum_{p,q} |I_p||J_q| = \sum_p |I_p| \sum_q |J_q|$. We can now take infima on the right to complete the proof.

Show this: if either of A, B is a set of measure zero, so is $A \times B$. This is one reason for using the "$0 \times \infty = 0$" convention.

Theorem: *For any measurable sets* $E \subseteq \mathbb{R}^n$ *and* $M \subseteq \mathbb{R}^m$, $E \times M$ *is measurable and* $|E \times M| = |E||M|$.

Proof: Assume $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ are both compactand $|L| > 0$. These assumptions won't be fully used below. What we need to know is that Cartesian products of certain sets are measurable in $\mathbb{R}^{n+m}$. Thus products of pairs of open or closed sets are measurable, as well as the product of an open set and a closed set, since the open set can be written as a countable union of closed sets. Measurability of other products will follow from σ-algebra properties unless otherwise stated.

So $K \times L$ is closed and bounded, hence measurable, with finite measure. Thus $|K \times L| \le |K||L|$.
To show: $|K \times L| \ge |K||L|$. There are open sets $G \subseteq \mathbb{R}^n$ and $\mathcal{O} \subseteq \mathbb{R}^m$ such that $K \subseteq G$, $L \subseteq \mathcal{O}$ and $|G \setminus K| < \alpha$, $|\mathcal{O} \setminus L| < \beta$, where α and β are positive numbers we can choose later once we figure out how. Then $(K \times L) \subseteq (G \times \mathcal{O})$ and

$$(G \times \mathcal{O}) \setminus (K \times L) = \big((G \setminus K) \times L\big) \cup \big(K \times (\mathcal{O} \setminus L)\big) \cup \big((G \setminus K) \times (\mathcal{O} \setminus L)\big)$$

(you should check this equality!!) so

$$|(G \times \mathcal{O}) \setminus (K \times L)| \le |(G \setminus K) \times L| + |K \times (\mathcal{O} \setminus L)| + |(G \setminus K) \times (\mathcal{O} \setminus L)| < \alpha|L| + |K|\beta + \alpha\beta.$$

Hence $|(G \times \mathcal{O})| \le |(K \times L)| + \alpha|L| + |K|\beta + \alpha\beta$. We write $G = \bigcup_p I_p$, $\mathcal{O} = \bigcup_q J_q$, non-overlapping unions of intervals in $\mathbb{R}^n$ and $\mathbb{R}^m$. Then $G \times \mathcal{O} = \bigcup_{p,q} I_p \times J_q$ is a non-overlapping union of intervals in $\mathbb{R}^{n+m}$,
so $|G \times \mathcal{O}| = \sum_{p,q} |I_p||J_q| \ge |K||L|$. Thus $|K \times L| > |K||L| - (\alpha|L| + |K|\beta + \alpha\beta)$.
Since α, β were arbitrary, $|K \times L| \ge |K||L|$, so actually $|K \times L| = |K||L|$.

Suppose now that K is just closed, and L is still compact. Then $K \times L$ is still closed, hence measurable. Then $|K \times L| = \lim_{p \to \infty} |K_p \times L|$, where the K_p are compact and increase to K. But then $|K \times L| = \lim_{p \to \infty} |K_p||L| = |K||L|$.
We could have worked with a closed L and L_p's at the same time...

We can show likewise that $|K \times L| = |K||L|$ if K is an F_σ. Then, if E is measurable, there is an F_σ set $H \subseteq E$ such that $|H| = |E|$. Thus $|(E \times L) \setminus (H \times L)|_e = |(E \setminus H) \times L|_e = 0$, so $E \times L$ is measurable. But then $|E \times L| = |H \times L| = |H||L| = |E||L|$. We can now keep E fixed, and approach a measurable set M of positive measure from within by compact sets L_q to finally show that $E \times M$ is measurable and $|E \times M| = |E||M|$.

It is not necessarily true that, if $A \times M$ is measurable and M is measurable, then A is measurable. But we can show:
Lemma: *If* I *is an interval with positive measure and* $A \times I$ *is measurable, then* A *is measurable.*

At first, assume $|A|_e < \infty$. Then, given $\epsilon > 0$ there exists an open G such that $A \times I \subseteq G$ and $|G \setminus (A \times I)| < \epsilon|I|$.

As the first step, let's find an open set that includes A but is not "too big." For each $x \in A$, the function $\operatorname{dist}((x, y), G^c)$ is a positive continuous function on I that therefore has a positive minimum, $\delta(x)$. Hence $B_{\delta(x)}(x) \times I \subseteq G$. We define the set $\mathcal{O} := \bigcup_{x \in A} B_{\delta(x)}(x)$, which is an open set containing A. Moreover, $A \times I \subseteq \mathcal{O} \times I \subseteq G$.

For the next step, let's show that since $A \times I$ is measurable, then $|A \times I| = |A|_e|I|$. Returning to G and $\mathcal{O}$ we see that

$$|A|_e|I| + \epsilon|I| \geq |A \times I| + \epsilon|I| > |G| \geq |\mathcal{O} \times I| = |\mathcal{O}||I| > |A|_e|I|.$$

Since ϵ was arbitrary, the result holds.

Now for the last step: We have $(\mathcal{O} \times I) \setminus (A \times I) = (\mathcal{O} \setminus A) \times I$, *which is a measurable set!* Therefore

$$\epsilon|I| > |G \setminus (A \times I)| \geq |(\mathcal{O} \times I) \setminus (A \times I)| = |(\mathcal{O} \setminus A) \times I| = |\mathcal{O} \setminus A|_e|I|$$

and so $|\mathcal{O} \setminus A|_e < \epsilon$. Hence A is measurable. If $|A|_e = \infty$, then $A \times I = \bigcup_k (A \times I) \cap (B_k(0) \times I)$, and since these sets equal $(A \cap B_k(0)) \times I$, $A \cap B_k(0)$ is measurable for each k, and so A is measurable.

Proof of Theorem 5.1

To show that $R(f, E)$ is measurable if f is measurable, we use the construction in Theorem (4.13) of an increasing sequence of simple functions that converges to f, Cartesian products (to show $R(f_k, E)$ is measurable) and Lemma (5.3), which asserts that the graph of a measurable function has measure zero.

Next we suppose that the sets E and $R(f, E)$ are measurable. We want to show that $\{x \in E : f(x) > \alpha\}$ is measurable for all real α. If $\alpha < 0$ there is nothing to prove. When $\alpha \geq 0$, we define the set V_α by

$$V_\alpha := (R(f, E) - \alpha\, \mathbf{e}_{n+1}) \cap \mathbb{R}^{n+1}_+,$$

where $\mathbf{e}_{n+1}$ is the vector in $\mathbb{R}^{n+1}$ whose coordinates are all zero except the last, which is 1, and $\mathbb{R}^{n+1}_+$ consists of all vectors in $\mathbb{R}^{n+1}$ with positive last coordinate. Then V_α is measurable.

We will transform V_α into the Cartesian product of $\{x \in E : f(x) > \alpha\}$ and an interval. We will do the transforming in such a way that every step will produce a measurable set. Then we can use the preceding work on Cartesian products and measurability to deduce that $\{x \in E : f(x) > \alpha\}$ is measurable. We will need a countable collection of steps, and a final countable union.

We'll use the fact that a linear transformation maps measurable sets to measurable sets, applied to the transformation T_k defined for k a positive integer by

$$T_k(x,\, y) := (x,\, ky).$$

It is routine to check that the sets $T_k V_\alpha$ increase with k. **Claim:** their union, V, is $\{x \in E : f(x) > \alpha\} \times (0, \infty)$. If $(x,\, y) \in V$, there exists k such that $(x,\, y) \in T_k V_a$, so $(x,\, y) = (x,\, k\eta)$ for some $(x,\, \eta) \in V_\alpha$. By construction $\eta > 0$ and $(x,\, \eta + \alpha) \in R(f, E)$. Hence $\eta + \alpha \leq f(x)$. On the other hand, if $f(x) > \alpha$ and $y > 0$ then for some k we have $f(x) > \alpha + (y/k)$, or $f(x) - \alpha > y/k$, so that $(x,\, y/k) \in V_\alpha$, which implies that $(x,\, y) \in V$. Thus $V = \{x \in E : f(x) > \alpha\} \times (0, \infty)$ is measurable. Hence $\{x \in E : f(x) > \alpha\} \times [1,\, 2]$ is measurable.

As we have shown, $\{x \in E : f(x) > \alpha\}$ is therefore measurable, as desired.