

Multi-index notation

A **multi-index**, or **multi-order**, is an element α of \mathbf{N}^n . That is, α is an n -tuple of non-negative integers: $\alpha = (\alpha_1, \dots, \alpha_n)$.

The **order**, or **length**, of α is $|\alpha| := \alpha_1 + \dots + \alpha_n$.

If α and β are multi-indices, we say that $\alpha \leq \beta$ if $\alpha_k \leq \beta_k$ for $k = 1, \dots, n$.

If α is a multi-index, $\alpha! := \alpha_1! \cdots \alpha_n!$. This is read "alpha factorial."

If α and γ are multi-indices, $\binom{\alpha}{\gamma} := \frac{\alpha!}{(\alpha - \gamma)! \gamma!}$; this is read "alpha choose gamma."

If x is in \mathbf{R}^n , and α is a multi-index, $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We read this as "x to the alpha (power)."

We use multi-orders to express the operator of (mixed) partial differentiation. Two fairly common notations we will use are

$$\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha \text{ and } \mathbf{D}^\alpha \text{ to denote } \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

We also express $\mathbf{D}^\alpha f$ as $\mathbf{f}^{(\alpha)}$. A wide variety of notations is used for differentiation.

A **polynomial of degree d** in $x \in \mathbf{R}^n$ is a function of the form $P(x) := \sum_{|\alpha| \leq d} c_\alpha x^\alpha$,

where the **coefficients** c_α are complex numbers and d is in \mathbf{N} , and at least one c_α with $|\alpha| = d$ is non-zero.

We will use the notations P_α and \bar{P}_α , respectively, to denote the **multiplication**

operators defined by $\mathbf{P}_\alpha \mathbf{f}(x) := (ix)^\alpha f(x)$ and $\bar{\mathbf{P}}_\alpha \mathbf{f}(x) := (-ix)^\alpha f(x)$.

Exercise: Prove that, for d in \mathbf{N} , $(x_1 + \dots + x_n)^d = \sum_{|\alpha| = d} \frac{d!}{\alpha!} x^\alpha$. This is also known as a "multinomial theorem."

Exercise: Prove **Leibniz' Rule:** $(fg)^{(\alpha)} = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} f^{(\gamma)} g^{(\alpha - \gamma)}$. Assume all the indicated partial derivatives are continuous functions.

Exercise: Determine how many multi-indices α there are, for a given n , with $|\alpha| = d$.

Exercise: Find the formula for $(x + y)^\alpha$ in terms of powers x^β and y^γ . Here, x and y are in \mathbf{R}^n .

Let \mathcal{S} denote the (Laurent) **Schwartz class** of "rapidly decreasing functions," defined to be the set of all C^∞ complex-valued functions $f(x)$ defined on \mathbf{R}^n such that, for all multi-indices α and multi-orders β , $x^\alpha f^{(\beta)}(x)$ is a bounded function of x . The bound can depend on α and β .

Theorem: $\mathcal{S} \subseteq L^1(\mathbf{R}^n) \cap C_0(\mathbf{R}^n)$.

Proof: Since $(1 + |x|^2)^n$ is a polynomial, and $(1 + |x|^2)^n f(x)$ is bounded if f is in \mathcal{S} , the result follows at once because $(1 + |x|^2)^{-n}$ is integrable and in $C_0(\mathbf{R}^n)$.

Theorem: \mathcal{S} is a complex linear space that is closed under partial differentiations of all orders, multiplication by polynomials, and pointwise products (so that \mathcal{S} is an algebra).

Proof is straightforward, and uses Leibniz' Rule.

Let M denote the set of all C^∞ complex-valued functions $m(x)$ defined on \mathbf{R}^n such that, for each multi-order α , $m^{(\alpha)}(x)$ is bounded by a polynomial in x . The bound can depend on α .

Theorem: $M\mathcal{S} \subseteq \mathcal{S}$.

Proof is straightforward, and uses Leibniz' Rule.

Lemma (on integration by parts): If f and g are in $M \cup \mathcal{S}$, and at least one of them is in \mathcal{S} , then for $k = 1, \dots, n$,
$$\int \frac{\partial f}{\partial x_k} g \, dx = - \int f \frac{\partial g}{\partial x_k} \, dx.$$

Proof: Without loss of generality, we may assume $k = n$. Using Fubini's theorem we write the integral as an iterated integral with the dx_n integral performed first. If we prove the theorem for $n = 1$, the rest now follows. Since the integral is the limit, as $R \rightarrow +\infty$, of the integrals over $(-R, R)$, we can use ordinary integration by parts on each such integral, and note that the boundary terms tend to 0, by the previous theorem.

The following lemma *could* be phrased in terms making little use of Lebesgue theory. Since we have it, though, we will use it, and probably achieve superfluous generality.

Lemma (on differentiation under the integral sign): Let Y be a measure space, with measure μ , and suppose that

- (i) $F(x, y)$ is measurable on $I \times Y$, where I is an open interval contained in \mathbf{R} ;
- (ii) for each x in I , $F(x, y)$ is an integrable function of y ;
- (iii) for a.e. y in Y , $F(x, y)$ is differentiable in I ;
- (iv) there exists a function g in $L^1(d\mu)$ such that, for all x in I , and a.e. y in Y ,
$$\left| \frac{\partial F}{\partial x}(x, y) \right| \leq g(y).$$

Then, for all x in I ,
$$\frac{d}{dx} \int_Y F(x, y) \, d\mu(y) = \int_Y \frac{\partial F}{\partial x}(x, y) \, d\mu(y).$$

Scholium: The equation implicitly states that the derivative exists!

Proof of the lemma amounts to setting up the difference quotient, applying the Mean Value Theorem for a.e. y , and using Lebesgue's Dominated Convergence Theorem.

The Fourier transformation and the Schwartz class

There is a natural affinity between Fourier transforms and \mathcal{S} . Our first objective is to show that the Fourier transformation is an invertible linear mapping on \mathcal{S} . The “affinity” referred to will become clear as we derive several formulas relating the composition of the Fourier transformation with operators involved in the definition of \mathcal{S} .

We define the Fourier transform, \hat{f} of f in \mathcal{S} to be the function given by

$$\hat{f}(\xi) := \int e^{-i\xi \cdot x} f(x) dx. \text{ We will sometimes write } Ff \text{ for } \hat{f}.$$

We have to show, for each f in \mathcal{S} , that $\xi^{\alpha} \hat{f}^{(\beta)}(\xi)$ is a bounded function of ξ .

The first step is to show that $\hat{\mathcal{S}} \subseteq C^{\infty}$.

Theorem: If $f \in \mathcal{S}$, then \hat{f} is a C^{∞} function.

Proof: We can apply the differentiation lemma with $Y = \mathbf{R}^n$, $d\mu(y) := |f(y)| dy$, and $F(\xi, y) := e^{-i\xi \cdot y} \operatorname{sgn} f(y)$. Recall that $\operatorname{sgn} z := z/|z|$ for a complex number $z \neq 0$; $\operatorname{sgn} 0 = 0$.

We find, inductively, that $D^{\alpha} \hat{f}(\xi) = \int e^{-i\xi \cdot y} (-iy)^{\alpha} f(y) dy$, or

$$\boxed{D^{\alpha} F = F P_{\alpha}}$$

Since f in \mathcal{S} implies $P_{\alpha} f$ is in \mathcal{S} , and \mathcal{S} is contained in L^1 , the desired result follows since the Fourier transform of an integrable function is continuous.

Theorem: If $f \in \mathcal{S}$, then for each multi-index α , $P_{\alpha} \hat{f}$ is a bounded function.

Proof: Since \mathcal{S} is closed under partial differentiations, we can prove the theorem by verifying the formula

$$\boxed{F D^{\alpha} = P_{\alpha} F}$$

We do this by applying the integration by parts lemma, inductively:

$\int e^{-i\xi \cdot x} D^{\alpha} f(x) dx = \int (i\xi)^{\alpha} e^{-i\xi \cdot x} f(x) dx$. The minus signs in the integrations by parts are cancelled.

Theorem: $F\mathcal{S} \subseteq \mathcal{S}$.

Proof: We have $\xi^{\alpha} \hat{f}^{(\beta)}(\xi) = \xi^{\alpha} ((-ix)^{\beta} f(x)) ^{\wedge}(\xi) = (-i)^{|\alpha|} (D^{\alpha} ((-ix)^{\beta} f(x))) ^{\wedge}(\xi)$, by the last two theorems (we have to juggle powers of $-i$, and abuse notation; $(-i)^{|\alpha|}$ “should” be $(-i)^{|\alpha|}$).

Next we prove the inversion theorem for \mathcal{S} , by using an approximate identity based on an important example of an \mathcal{S} function.

Example: $\int e^{-i\xi \cdot x} e^{-|x|^2/2} dx = (2\pi)^{n/2} e^{-|\xi|^2/2}$.

It is routine to verify that $e^{-|x|^2/2}$ is in \mathcal{S} , using the fact that a partial derivative of this function is a polynomial times $e^{-|x|^2/2}$. The formula is derived by Jones: FT16g, pp 301-302. We need a variant of this, that uses the formula for an affine change of variables:

$$\int e^{-i\xi \cdot x} e^{-\varepsilon|x|^2/2} dx = \left(\frac{2\pi}{\varepsilon}\right)^{n/2} e^{-|\xi|^2/2\varepsilon}.$$

The change of variables is $Tx = x/\sqrt{\varepsilon}$. The general formula in this case is

$$f(Tx) \wedge (\xi) = \varepsilon^{-n/2} \hat{f}(\varepsilon^{1/2}\xi).$$

The Inversion Formula we want is

$$f(x) = (2\pi)^{-n} \int e^{i\xi \cdot x} \hat{f}(\xi) d\xi.$$

We cannot replace $\hat{f}(\xi)$ by the integral and change the order of integration. But we can take the limit, as $\varepsilon \rightarrow 0$, of $f_\varepsilon(x) := (2\pi)^{-n} \int e^{-\varepsilon|\xi|^2/2} e^{i\xi \cdot x} \hat{f}(\xi) d\xi$, and show that it is $f(x)$. In the last formula, we *can* change the order of integration:

$$\begin{aligned} f_\varepsilon(x) &:= (2\pi)^{-n} \int e^{-\varepsilon|\xi|^2/2} e^{i\xi \cdot x} \left(\int e^{-i\xi \cdot y} f(y) dy \right) d\xi \\ &= (2\pi)^{-n} \int \left(\int e^{-\varepsilon|\xi|^2/2} e^{-i\xi \cdot (y-x)} d\xi \right) f(y) dy \\ &= \int (2\pi\varepsilon)^{-n/2} e^{-|y-x|^2/2\varepsilon} f(y) dy. \end{aligned}$$

This is one of the approximate identity formulas. Since f is continuous and bounded, we can apply the substitute result for L^∞ (see Jones, Ch 12 # 17). In our case, the approximate identity has integral 1, so $f_\varepsilon(x)$ converges to $f(x)$, as desired. And the integral that defines $f_\varepsilon(x)$ converges to the integral on the right in the Inversion Formula.

We have proved:

Theorem: F is invertible on \mathcal{S} , and $F^{-1}f(x) = (2\pi)^{-n} (Ff)(-x)$.

Some applications of the Fourier Inversion Theorem for \mathcal{S} .

1). **The Fourier transformation is one-to-one on \mathcal{S} .** This follows from the invertibility.

It is tempting to try to prove this without using the inversion theorem.

2). Let us define some semi-norms on \mathcal{S} .

Definition: For f in \mathcal{S} , and α and β in \mathbf{N}^n , let $\|f\|_{\alpha\beta} := \sup_{x \in \mathbf{R}^n} |x^\alpha f^{(\beta)}(x)|$.

Theorem: For each α and β in \mathbf{N}^n , $\|f\|_{\alpha\beta}$ is a norm on \mathcal{S} .

To see this, we note that $\|f\|_{\alpha\beta} = 0$ implies that $x^\alpha f^{(\beta)}(x)$ is identically 0. Since x^α is 0 in a closed set of measure 0, and $f^{(\beta)}(x)$ is continuous, we conclude that $f^{(\beta)}(x)$ is identically 0. Then, since $F D^\beta = P_\beta F$, we conclude, as before, that the Fourier transform of f is identically 0. It follows that f is identically 0, by application 1).

3). **Theorem:** \mathcal{S} is closed under convolution.

Proof: This can be shown directly from the definition of \mathcal{S} . To show that $\|f * g\|_{\alpha\beta} < \infty$ we can use the differentiation lemma to differentiate, say, f . Then we write

$$\begin{aligned} x^\alpha (f * g)^{(\beta)}(x) &= x^\alpha (f^{(\beta)} * g)(x) = x^\alpha \int f^{(\beta)}(x - y) g(y) dy = \\ &= \int (x - y + y)^\alpha f^{(\beta)}(x - y) g(y) dy = \int (x - y + y)^\alpha f^{(\beta)}(x - y) g(y) dy. \end{aligned}$$

This is equal to a sum of terms of the form $\text{const.} \int (x - y)^\alpha f^{(\beta)}(x - y) y^\gamma g(y) dy$. We recognize that each of these is the convolution of two integrable functions, hence is bounded.

We can also prove this using the inversion theorem and the equation $\int \hat{f}(\xi) g(\xi) d\xi = \int f(x) \hat{g}(x) dx$, valid for pairs f, g of integrable functions. Suppose f and g are in \mathcal{S} .

Then $f * g(x) = \int f(x - y) g(y) dy = (2\pi)^{-n} \int f(x - y) \hat{\hat{g}}(-y) dy = (2\pi)^{-n} \int f(x + y) \hat{\hat{g}}(y) dy = (2\pi)^{-n} \int f(x + \bullet) \hat{(\hat{g})}(\eta) d\eta = (2\pi)^{-n} \int e^{i\eta \cdot x} \hat{f}(\eta) \hat{\hat{g}}(\eta) d\eta$. We recognize from this equation that $f * g$ is the Fourier transform of a function in \mathcal{S} , namely the product of the two functions \hat{f} and $\hat{\hat{g}}$, and hence $f * g$ is in \mathcal{S} .

4). **A formula for the Fourier transform of a product.**

If f and g are in \mathcal{S} , then $\boxed{F f g = (2\pi)^{-n} F f * F g.}$

To show this, we use the second proof that \mathcal{S} is closed under convolution, with f and g replaced by their Fourier transforms. In the last member of the equations we recognize the Fourier inversion formula.

5). Parseval's formula for \mathcal{S} .

$$\int |f(x)|^2 dx = (2\pi)^{-n} \int |\hat{f}(\xi)|^2 d\xi.$$

This is an application of 4), and recognition that the integral of a function is the value of its Fourier transform at 0. To carry it out we also need a formula for the Fourier transform of the complex conjugate of a function in terms of its Fourier transform. Let $Cf(x) := \overline{f(x)}$. This is not a linear map, but is *conjugate* linear. Such mappings have the same continuity properties as linear ones. Let $Rf(x) := f(-x)$. Then $RC = CR$, and $\overline{FR} = R\overline{F}$, but

$$\int e^{-i\xi \cdot x} \overline{f(x)} dx = \overline{\int e^{i\xi \cdot x} f(x) dx} = \overline{\hat{f}(-\xi)}.$$

That is, $FC = RCF$. Then, in 4), $FfCf = (2\pi)^{-n} Ff * FCF = (2\pi)^{-n} Ff * RCFf$. That is, $\int e^{-i\xi \cdot x} |f(x)|^2 dx = (2\pi)^{-n} \int \hat{f}(\xi - \eta) \overline{\hat{f}(-\eta)} d\eta = (2\pi)^{-n} \int \hat{f}(\xi + \eta) \overline{\hat{f}(\eta)} d\eta$. Finally, we set $\xi = 0$ to get the desired result. This result says that, when properly normalized, the Fourier transformation is an isometry with respect to the L^2 norm on \mathcal{S} .

Some applications of these properties of \mathcal{S} outside \mathcal{S} .

1. **The Fourier Inversion Theorem on L^1 .** If f and \hat{f} are both in $L^1(\mathbf{R}^n)$, then f is equal almost everywhere to a continuous function, and for almost all x ,

$$f(x) = (2\pi)^{-n} \int e^{i\xi \cdot x} \hat{f}(\xi) d\xi.$$

This says that if the Fourier transform of f is integrable, then f is the Fourier transform of an integrable function. The set of all such functions is an algebra with respect to both convolution and pointwise multiplication.

Proof 1 (outline): Examine the proof for \mathcal{S} , and note that all we used there was an approximate identity in \mathcal{S} and the fact that, for f in \mathcal{S} , \hat{f} is in L^1 . Here, we assume that \hat{f} is in L^1 . In fact, we can use approximate identities that are not in \mathcal{S} .

Proof 2: Since \mathcal{S} contains the infinitely differentiable functions with compact support, \mathcal{S} is dense in L^1 . Thus there exists a sequence $f_k \rightarrow f$, f_k in \mathcal{S} . As in the proof of the completeness of L^p , a subsequence exists that converges pointwise, a.e. Without loss (by renumbering) we may assume the sequence converges both pointwise a.e. and in L^1 . Then $\hat{f}_k - \hat{f}$ converges uniformly to zero.

By the inversion theorem,

$$(2\pi)^{-n} \hat{f}_k(-x) - (2\pi)^{-n} \hat{f}(-x) \text{ converges a.e. to } f(x) - (2\pi)^{-n} \hat{f}(-x).$$

Thus, for any C^∞ function g with compact support, $\int \hat{g}(\hat{f}_k - \hat{f}) \rightarrow 0$.

Hence $\int g(\hat{f}_k - \hat{f}) \rightarrow 0$. That is, $\int g((2\pi)^n f - \hat{f}) = 0$ for all such g . In any ball there is a sequence of such g 's that converges pointwise a.e. to $\text{sgn}((2\pi)^n f - \hat{f})$.

Thus $\int_{|x| < R} |(2\pi)^n f - \hat{f}| dx = 0$ for all R , so $f = (2\pi)^{-n} \hat{f}$, as desired.

Remark: Four sentences in the proof, beginning with "Thus, for any C^∞ function g ," comprise a very useful tool that deserves to be emphasized. One consequence is:

Theorem: If f is locally integrable, then for $1 \leq p \leq \infty$,

$$\|f\|_p = \sup \left\{ \int f g : g \in C_c^\infty, \text{ and } \|g\|_{p'} \leq 1 \right\}.$$

The proof is essentially covered in those three sentences.

Example: The Fourier transform of $e^{-|x|}$ is $\frac{2}{1 + \xi^2}$. This is an integrable function. By the inversion theorem, its Fourier transform is $2\pi e^{-|x|} = 2\pi e^{-|x|}$. That is,

$$\int \frac{e^{-i\xi x}}{1 + x^2} dx = \pi e^{-|\xi|}.$$

2. Definition of the Fourier transformation on L^2 . Parseval's formula for \mathcal{S} can be rewritten $\int |\hat{f}(\xi)|^2 d\xi = (2\pi)^n \int |f(x)|^2 dx$. Parseval's formula shows that F is continuous on \mathcal{S} with respect to the L^2 norm. Let us define an operator F_2 on L^2 by a procedure known as "extension by continuity from a dense subset." We will normalize the operator so that F_2 becomes an isometry. Let $f \in L^2$. Since \mathcal{S} is dense in L^2 , there exists a sequence $f_k \rightarrow f$, f_k in \mathcal{S} . The convergence is in L^2 . Then $(2\pi)^{-n/2} F f_k$ is a Cauchy sequence in L^2 . If some other sequence $g_k \rightarrow f$, g_k in \mathcal{S} , the sequence $(2\pi)^{-n} F (f_k - g_k)$ converges to 0 in L^2 . Thus

$$F_2 f := L^2\text{-}\lim_{k \rightarrow \infty} (2\pi)^{-n/2} F f_k$$

is well defined, and $\|F_2 f\|_2 = \lim_{k \rightarrow \infty} \|(2\pi)^{-n/2} F f_k\|_2 = \lim_{k \rightarrow \infty} \|f_k\|_2 = \|f\|_2$.

This argument shows also that F_2 is invertible, with $F_2^{-1} = R F_2$.

Example: The Fourier transform of the characteristic function of $(-1, 1)$ is $\frac{2 \sin \xi}{\xi}$. We

have, since this last function is in L^2 , that $\frac{2}{\pi} \int \frac{\sin^2 \xi}{\xi^2} d\xi = 2$, by the isometric property of

F_2 , or $\int \frac{\sin^2 \xi}{\xi^2} d\xi = \pi$.

3. The Fourier transform of a product.

If f and g are in L^2 , or if f and g are in L^1 , and the Fourier transform of one of them is in L^1 , then $Ffg = (2\pi)^{-n} Ff * Fg$.

In either case, the product fg is integrable. If $f_k \rightarrow f$, f_k in \mathcal{S} , and $g_k \rightarrow g$, g_k in \mathcal{S} , the convergence taking place in L^2 , it follows from $Ff_k g_k = (2\pi)^{-n} Ff_k * Fg_k$ that the desired result holds. In the second case we take $f_k \rightarrow f$, f_k in \mathcal{S} , $h_k \rightarrow (2\pi)^{-n} \widehat{Rg}$, h_k in \mathcal{S} , the convergence of both sequences taking place in L^1 . Then $\widehat{h_k} \rightarrow g$, uniformly. In the equation $Ff_k \widehat{h_k} = (2\pi)^{-n} Ff_k * F\widehat{h_k} = Ff_k * Rh_k$, $f_k \widehat{h_k}$ converges, in L^1 , to fg , Ff_k converges uniformly to Ff , and Rh_k converges in L^1 to $(2\pi)^{-n} \widehat{g}$. Therefore both sides converge uniformly to the desired equation.

Exercise: Find the convolution of $\frac{1}{a^2 + x^2}$ and $\frac{1}{b^2 + x^2}$, where $a > 0, b > 0$.