Some miscellaneous definitions, theorems and lemmas

Theorem: If u is a tempered distribution, in $\mathcal{S}'(\mathbf{R})$, and $u' = 0$, then $u = c$, for some constant c, in the sense that, for all f in $\mathcal{S}(\mathbf{R})$, $(u, f) = c\int f(x)\,dx$.

Proof: The proof will show more, namely, that every tempered distribution has a primitive, and that any two primitives differ by a constant. We begin by constructiong a "near-primitive," in $\mathcal{S}$, for functions f in $\mathcal{S}$. Let

$$Uf(x) := \int_{-\infty}^{x} f(t)\,dt - \left(\int f(t)\,dt\right)\int_{-\infty}^{x} W(t)\,dt, \text{ where } W(x) := \frac{e^{-x^2}}{\sqrt{\pi}}\ .$$

The function W is sometimes called the **Weierstrass kernel**.

We have $\frac{d}{dx}\,Uf(x) = f(x) - \left(\int f(t)\,dt\right) W(x)$, so $\frac{d}{dx}\,Uf(x) \in \mathcal{S}$.

Note that $Uf(x) \to 0$ as $|x| \to \infty$. For every positive integer n, $x^n\,Uf(x) = \dfrac{Uf(x)}{x^{-n}}$, for x not 0. We can apply the rule of l'Hospital; the limit of this ratio is equal to that of

$$\frac{f(x) - \left(\int f(t)\,dt\right) W(x)}{(-n)x^{-n-1}}\ ,$$

in case it exists. The limit does exist, as $|x| \to \infty$, and it is 0. It follows that Uf is in $\mathcal{S}$. We will also need to know that U is a continuous map on $\mathcal{S}$. Let us take that for granted for the moment.

Now let u be in $\mathcal{S}'$. Define Pu by $(Pu, f) := -(u, Uf)$. This is, modulo what we are taking for granted, a tempered distribution. Let us calculate the distribution derivative of Pu. By definition, $(\frac{d}{dx} Pu, f) = -(Pu, \frac{df}{dx}) = -(u, -U\frac{df}{dx}) = (u, f)$, because $\int \frac{df}{dx}\,dt = 0$. Thus $\frac{d}{dx}\,Pu = u$. That is, they agree for all f in $\mathcal{S}$. Suppose now that there is also a primitive V for u. Then, with $w := Pu - V$, $(\frac{d}{dx}\,w, f) = 0$ for all f in $\mathcal{S}$. In particular, $(\frac{d}{dx}\,w, Uf) = 0$ for all f in $\mathcal{S}$. This implies that $(w, \frac{d}{dx}\,Uf) = 0$ for all f in $\mathcal{S}$, or $(w, f - \left(\int f(t)\,dt\right) W) = 0$. Therefore $(w, f) = \left(\int f(t)\,dt\right)(w, W)$. But $C := (w, W)$ is a constant. Thus, $(w, f) = \int C\,f(t)\,dt$ for all f in $\mathcal{S}$, and this is what we mean by "$w = C$."

Now we need a **lemma:** To show that U is a continuous map on $\mathcal{S}$. Recall that we must find an estimate for $\|Uf\|_{\alpha\beta}$, in terms of a finite number of norms $\|f\|_{\gamma\delta}$, for each (α, β) in $\mathbf{N}\times\mathbf{N}$.
If $\beta > 0$, $\|Uf\|_{\alpha\beta} = \|f - cW\|_{\alpha,\beta-1}$, where $c = \left(\int f(t)\,dt\right)$. This can be estimated in the required manner, as we have seen before. The estimate will involve estimates for $\|f\|_1$, and we have seen how to estimate this: $\|f\|_1 \le \|f\|_{00} + \|f\|_{2,0}$. Thus $\|Uf\|_{00}$ causes no difficulties either. So all we have to do is estimate $\|Uf\|_{n0}$, when $n > 0$. If $|x| \le 1$,

$| x^n Uf(x) | \le | Uf(x) | \le \| f \|_1(1 + \| W \|_1) = C\| f \|_1$. If $x < -1$, we use the estimates $|t^{n+1} f(t)| \le \| f \|_{n+1,0}$ and $|t^{n+1} W(t)| \le \| W \|_{n+1,0}$ to see that

$$| x^n Uf(x) | \le |x|^n \int_{-\infty}^{x} |f(t)|\, dt + \left(\int |f(t)|\, dt\right) |x|^n \int_{-\infty}^{x} W(t)\, dt \le$$

$$|x|^n \int_{-\infty}^{x} \frac{\| f \|_{n+1,0}}{|t|^{n+1}}\, dt + \| f \|_1\, |x|^n \int_{-\infty}^{x} \frac{\| W \|_{n+1,0}}{|t|^{n+1}}\, dt = \frac{\| f \|_{n+1,0}}{n} + \| f \|_1 \frac{\| W \|_{n+1,0}}{n} .$$

This is a suitable estimate; the lemma follows, since the case $x > 1$ is similarly treated.

Here is an alternate approach to proving the estimate for $\| Uf \|_{n0}$, when $n > 0$. According to the Cauchy Mean value theorem, with y either much larger than x, or much smaller than x,

$$\frac{Uf(x)}{x^{-n}} - \frac{Uf(y)}{y^{-n}} = \frac{f(z) - \left(\int f(t)\, dt\right) W(z)}{(-n)z^{-n-1}} \text{ , for some } z \text{ strictly between } x \text{ and } y.$$

The right-hand side, being $z^{n+1}\dfrac{f(z) - \left(\int f(t)\, dt\right) W(z)}{(-n)}$, is |dominated| by

$(\| f \|_{n+1,0} + \| f \|_1 \| W \|_{n+1,0})\, n^{-1}$. Now, with absolute values on the left, let y tend to either infinity. An estimate of the desired form follows from this one.

Definition: The support, **supp u**, of a distribution u is the *complement* of the largest open set Ω with this property: for every f in S such that $\text{supp } f \subseteq \Omega$, $(u, f) = 0$.

Theorem: If $\text{supp } u = \{0\}$, then u is a linear combination of the Dirac δ-function and its derivatives.

Reminder: a linear combination is always finite!

Proof: Because u is a tempered distribution, there exist N in $\mathbf{N}$ and constants $C_{\alpha\beta}$, for $|\alpha| \le N$, $|\beta| \le N$, such that for all f in S, $|(u, f)| \le \sum_{|\alpha|\le N,\, |\beta|\le N} C_{\alpha\beta} \| f \|_{\alpha\beta}$. We are going to express an arbitrary function f in S, near 0, as a polynomial times a "patch function," plus a remainder. We will use Taylor's Theorem in n variables. Then we'll show that the remainder is in the kernel of u. The last step yields the linear combination, as in the proof of the previous theorem.

the "patch" function

We know that there is a C^∞ function $p(t)$ of one real variable such that $0 \le p(t) \le 1$ for all t, $p(t) = 1$ for all t with $|t| \le 1/2$, and $p(t) = 0$ for $|t| > 1$. We define a one-parameter family of "patch functions," $p_\varepsilon(x) := p(|x|/\varepsilon)$. Note that p_ε is C^∞ and has support contained in the ball of radius ε, and center 0.

Now, for all f in S, and all $\varepsilon > 0$, $h_\varepsilon(x) := p_\varepsilon(x) f(x)$ is also in S, and $(u, h_\varepsilon) = (u, f)$. This is true because the support of $f - h_\varepsilon$ does not contain 0. We can also use this idea to show that, if f and g

are in S, and $f(x) = g(x)$ for $|x| < \varepsilon$, then $(u, f) = (u, g)$. Taylor's Theorem in n variables gives us the representation, near 0, of f in S as the sum of a polynomial $T(x)$ of degree at most N, say, and a remainder $R(x)$ such that $R^{(\beta)}(x) = O(|x|^{N+1-|\beta|})$ for each β with $|\beta| \le N$. Moreover, $R^{(\beta)}(x)$ is the remainder for $f^{(\beta)}(x)$ near 0, for each β with $|\beta| \le N$.

Taylor's Theorem in n variables is derived from the one-dimensional theorem by way of the chain rule and a counting argument. We define $g(t) := f(x + ty)$. Then it can be shown by induction that $g^{(m)}(t) = \sum_{|\beta|=m} \frac{m!}{\beta!} D^\beta f(x + ty) y^\beta$. The induction step is a little tricky. For each j such that $\beta_j > 0$, We write $\beta = \beta - e_j + e_j$. Then we multiply and divide the term $\frac{m!}{(\beta - e_j)!}$ by β_j.

When we apply Taylor's Theorem to g, we get $g(t) = \sum_{m=0}^{M} \frac{g^{(m)}(0)}{m!} t^m + R_M(t)$, where M is in **N**.

The remainder, $R_M(t)$, is usually expressed in one of the two forms

$$R_M(t) = \frac{g^{(M+1)}(s)}{(M+1)!} t^m, \text{ where } s \text{ is strictly between } 0 \text{ and } t,$$

or

$$R_M(t) = \int_0^t \frac{(t-s)^M}{M!} g^{(M+1)}(s)\, ds.$$

Finally, we set $t = 1$, and express $g^{(m)}(0)$ in terms of f. This gives Taylor's Theorem for functions of n variables, in L. Schwartz'(?) notation:

$f(x + y) = \sum_{|\beta| \le M} \frac{D^\beta f(x)}{\beta!} y^\beta + R_M(y)$, where the remainder, $R_M(y)$, is usually expressed in one of the two forms

$$R_M(y) = \sum_{|\beta|=M+1} \frac{D^\beta f(x + ty)}{\beta!} y^\beta, \text{ where } 0 < t < 1,$$

or

$$R_M(t) = \int_0^t (M+1)(t-s)^M \sum_{|\beta|=M+1} \frac{D^\beta f(x + sy)}{\beta!} y^\beta\, ds.$$

We now write $f(x) = T(x) + R(x)$, where T is the Taylor polynomial involving all the partial derivatives of f of order $\le N$, and R is the remainder, and we let $f_\varepsilon(x) := p_1(x)\, T(x) + p_\varepsilon(x)\, R(x)$. Since $f_\varepsilon(x) = f(x)$ for all x with $|x| < \varepsilon$, $(u, f_\varepsilon) = (u, f)$. Let us examine $(u, p_\varepsilon R)$, and show it must be 0. Now it is time to use the continuity of u:

$$|(u, p_\varepsilon R)| \le \sum_{|\alpha|\le N,\, |\beta|\le N} C_{\alpha\beta} \,\| p_\varepsilon R \|_{\alpha\beta}.$$

If $\beta \ne 0$, $D^\beta p_\varepsilon R$ is, by Leibniz' Rule, a sum of terms of the form

$$(D^{\beta-\gamma} p_\varepsilon)(D^\gamma R) = \varepsilon^{-|\beta|+|\gamma|} (D^{\beta-\gamma} p_1)(x/\varepsilon)\, O(|x|^{N+1-|\gamma|})$$
$$\le C\varepsilon^{-|\beta|+|\gamma|+N+1-|\gamma|} = C\varepsilon^{N+1-|\beta|} \le C\varepsilon.$$

Here, C can be taken to be the larger of: the largest of the constants involved in the estimates $O(|x|^{N+1-|\gamma|})$ for $R(x)$, and the largest of the quantities $\| p_1 \|_{0\gamma}$, for $\gamma \le \beta$. If $\alpha \neq 0$, the estimate is only improved. As we have argued before, $(u, p_\varepsilon R)$ is independent of positive ε, so $(u, p_\varepsilon R)$ is indeed 0, as desired.

Now we have $(u, f) = (u, p_1 T) + (u , p_\varepsilon R) = (u, p_1 T) = \sum_{|\beta|\le N} \frac{D^\beta f(0)}{\beta!} (u, x^\beta p_1 T)$.

Some abuse of notation has taken place; x^β really “should” be replaced by a symbol that does not involve the variable x. But it would be cumbersome to introduce notation, or, worse yet, represent x^β in terms of the multiplication operator P_α.

Definition: Suppose $u \in \mathcal{S}'$. Then, for any $h \in \mathbf{R}^n$, we define the **translate of u by h** by the equation $(\tau_h u, f) := (u, \tau_{-h} f)$, for all f in $\mathcal{S}$.

Definition: Suppose $u \in \mathcal{S}'$, and $p \in \mathbf{R}^n$. Then **u is periodic, of period p**, if $\tau_p u = u$.

Theorem: If $u \in \mathcal{S}'$ is periodic of period $p \in \mathbf{R}^n$, then the support of $\hat{u}$ is contained in ?

Proof: We have $(\hat{u} , f) = (u, \hat{f}) = (\tau_p u, \hat{f}) = (u, \tau_{-p}\hat{f}) = (u, (e^{ip\bullet x}f(x))^\wedge) = (\hat{u} , e^{ip\bullet x}f(x))$.
That is, $(\hat{u} , f) = (\hat{u} , e^{ip\bullet x}f(x))$. The same is true for $-p$. This means that $(\hat{u} , (e^{ip\bullet x} - 2 + e^{-ip\bullet x})f) = -4 (\hat{u} , \sin^2((p\bullet x)/2) f) = 0$. The function $\sin^2((p\bullet x)/2)$ is 0 whenever $p\bullet x = 2\pi n$ for some n in **Z**. This occurs in the union Π of a family of parallel hyperplanes. We will actually show that, if f is in S, and $f(x)$ and all $f^{(\beta)}(x)$ are 0 whenever x is in Π, then $(\hat{u}, f) = 0$. This will be done by showing that, if f is an $\mathcal{S}$-function of this sort (said to **vanish to all orders on** Π), then $\dfrac{f(x)}{\sin^2 \frac{p\bullet x}{2}}$ is in $\mathcal{S}$. The ratio is defined to be 0 in Π.