Math 2243, Fall 2002 Trial Solutions Using Leibniz' Rule

Introduction

A linear ordinary differential equation *with constant coefficients* and functions of the form e^{rx} go together very well!

We are concerned right now with second-order differential equations of the form $y'' + a_1y' + a_2y = F(x)$, where a_1 and a_2 are constants that are real numbers. But the methods discussed in this note apply to all linear ordinary differential equation with constant coefficients, at least in principle.

Background (can be skipped at first)

When we substitute $y = e^{rx}$ into our equation we get the equation

$$y'' + a_1y' + a_2y = r^2e^{rx} + a_1re^{rx} + a_2e^{rx} = P(r)e^{rx} = F(x),$$

where $P(r) = r^2 + a_1r + a_2$. If it were true that for some r_1 we had $F(x) = Ae^{r_1x}$, and we took $y_1 = Ce^{r_1x}$, then we'd get

$$y_1'' + a_1y_1' + a_2y_1 = Cr^2e^{r_1x} + a_1Cre^{r_1x} + a_2Ce^{r_1x} = CP(r_1)e^{r_1x} = Ae^{r_1x}$$

if we choose C so that $CP(r_1) = A$. If $P(r_1) \neq 0$ all we have to do is choose $C = A/P(r_1)$. If $P(r_1) = 0$ we're still OK if also $A = 0$, and then we could choose $C = 1$, or any other number. However, if $P(r_1) = 0$ and $A \neq 0$, this method will not work.

We can still find a solution, but we have to work harder. We need to have the idea of "linear differential operator." We need to know that if $\dfrac{\partial^2 f}{\partial x \partial y}$ is continuous, then $\dfrac{\partial^2 f}{\partial y \partial x}$ exists and that then $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$. To see what our application is, we let $H(x, r) := e^{rx}$, where x is a real number, and r is a complex number (thus r can be a real number, because the real numbers are just complex numbers with zero imaginary part). Then we have

$$\frac{\partial H}{\partial x} = rH \quad \text{and} \quad \frac{\partial H}{\partial r} = xH, \quad \text{by the Chain Rule.}$$

We can keep going, and get $\dfrac{\partial^2 H}{\partial r \partial x} = \dfrac{\partial}{\partial r}\dfrac{\partial H}{\partial x} = \dfrac{\partial}{\partial r}(rH) = H + r\dfrac{\partial H}{\partial r} = H + rxH = H + xrH = \dfrac{\partial^2 H}{\partial x \partial r}$. In general, we can see that we can take the derivatives in any order, without making any difference. We will think of H as a function $y(x)$ that has a "parameter" r. That lets us make sense of expressions like $\frac{\partial}{\partial r}y'' = \left(\frac{\partial}{\partial r}y\right)''$, that "really" means $\dfrac{\partial}{\partial r}\dfrac{\partial^2 H}{\partial x^2} = \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial H}{\partial r}\right)$, by just changing the order of differentiation. And yet, we can still think we are working in one variable; the r comes in "separately."

Another thing we do is recognize that differentiation is a linear operator, and (by analogy with matrix multiplication) we write $Dy = y'$, $D^2 = y''$, and so on.

We next write $y'' + a_1y' + a_2y = D^2y + a_1Dy + a_2y = D^2y + a_1Dy + a_2Iy = (D^2 + a_1D + a_2I)y =: L(y)$, so that L is a *linear differential operator with constant coefficients.* This is just a way of saying "multiply by a_2, add that to the result of differentiating and multiplying by a_1, and finally add the result of differentiating twice." This says "in advance" what we're going to do to y.

There is another thing we can do, that I did not mention in class: we can think of the operator L as a polynomial in D! That is, we can write $L(y) = P(D)y$. That is, $L(y) = y'' + a_1y' + a_2y = (D^2 + a_1D + a_2I)y = P(D)y$.

We know now that for any integer $k \geq 0$, $\left(\dfrac{\partial}{\partial r}\right)^k L(y) = L\left(\left(\dfrac{\partial}{\partial r}\right)^k y\right) = \left(\dfrac{\partial}{\partial r}\right)^k L(y)$ (we can work in either order). Right now, k will be one, two or three.

Let us write this in our new way: $\left(\dfrac{\partial}{\partial r}\right)^k [P(D)y] = \left[\left(\dfrac{\partial}{\partial r}\right)^k P(D)\right] y = \left[P(D)\left(\dfrac{\partial}{\partial r}\right)^k\right] y = P(D)\left[\left(\dfrac{\partial}{\partial r}\right)^k y\right]$.

We can always write this in the reverse order! When we apply this to $y(x) = e^{rx}$ we get, reversing the order,

$$P(D)\left[\left(\frac{\partial}{\partial r}\right)^k y\right] = P(D)\left[x^ke^{rx}\right] = \left(\frac{\partial}{\partial r}\right)^k \left[P(r)e^{rx}\right].$$

The Method You will need to read this with pencil and paper handy!
We want to avoid doing differentiations over and over. We use Leibniz' Rule:

$$\left(\frac{\partial}{\partial r}\right)^k [P(r)e^{rx}] = \sum_{\ell=0}^{k} \binom{k}{\ell} P^{(k-\ell)}(r)x^\ell e^{rx}.$$

When $k = 1$ this is

$$P(D)[xe^{rx}] = \left(\frac{\partial}{\partial r}\right)[P(r)e^{rx}] = P^{(1-0)}(r)x^0e^{rx} + P^{(1-1)}(r)x^1e^{rx} = P'(r)e^{rx} + P(r)xe^{rx}.$$

When $k = 2$:

$$P(D)[x^2e^{rx}] = \left(\frac{\partial}{\partial r}\right)^2 [P(r)e^{rx}] = P''(r)e^{rx} + 2P'(r)xe^{rx} + P(r)x^2e^{rx},$$

and when $k = 3$:

$$P(D)[x^3e^{rx}] = \left(\frac{\partial}{\partial r}\right)^3 [P(r)e^{rx}] = P'''(r)e^{rx} + 3P''(r)xe^{rx} + 3P'(r)x^2e^{rx} + P(r)x^3e^{rx}.$$

We'll do examples to illustrate the method.

Example 1 (Easy Case): Find the general solution of the ODE

$$y'' - 2y' - 3y = e^{2x}.$$

Here $P(D) = D^2 - 2D - 3I$ and the polynomial $P(r) = r^2 - 2r - 3$ has roots $r_1 = 1$ and $r_2 = -3$ by factoring. On the right-hand-side we have $F(x) = e^{ax}$ with $a = 2$. This value of a is not a root of P so we know that (for that reason!) $P(D)e^{2x} = P(2)e^{2x}$ and $P(2) = 4 - 4 - 3 = -3$, so we know that $P(D)e^{2x} = -3e^{2x}$. Therefore if we try $y_p = Ce^{2x}$ we get $P(D)y_p = CP(D)e^{2x} = -3Ce^{2x}$ because of linearity: we can factor the C out. Now we pick $C = -1/3$ and we get $y_p = (-1/3)e^{2x}$. In addition we know that the general solution of the homogeneous equation is $y_c = c_1e^x + c_2e^{-3x}$, so **the general solution is** $y(x) = c_1e^x + c_2e^{-3x} - (1/3)e^{2x}$.

This example is typical when $F(x) = Ae^{ax}$ and $P(a) \neq 0$. Thus when you are confronted by a second-order linear ODE with constant coefficients and the right-hand-side function is $F(x) = Ae^{ax}$, your first step is to check whether $P(a) = 0$. If $P(a) \neq 0$, then $y_p(x) = (A/P(a))e^{ax}$. We have not had to use "the method" here.

Example 2 (One root duplicated): Find the general solution of the ODE $y'' - 3y' - 2y = e^{2x}$. We have $P(r) = r^2 - 3r + 2 = (r - 1)(r - 2)$. Thus $P(a) = P(2) = 0$ and we cannot use the easy method. We *can* easily write down the general solution of the homogeneous equation: $y_c(x) = c_1e^x + c_2e^{2x}$. Since 2 is only a single root of $P(r)$, we will try $y_p(x) = Cxe^{2x}$.

Remark (can be skipped) If we try the easy method, we get $P(D)[Ce^{2x}] = 0$, not e^{2x}. But since $y = e^{2x}$ satisfies $y' = 2y$, we know that $y' - 2y = 0$, or that $(D - 2I)y = 0$. Therefore, if we *had* a valid y_p we would have $P(D)y_p = e^{2x}$. Then $(D - 2I)P(D)y_p = (D - 2I)e^{2x} = 0$. **In other words, our particular solution is a solution of a homogeneous linear ODE with constant coefficients.** Therefore the methods of Section 2.3 should work, even tho the homogeneous linear ODE with constant coefficients is third order. What are the roots of the auxilary polynomial? They are $r_1 = 1$, $r_2 = 2$ and $r_3 = 2$, so 2 is a double root, and we expect to use xe^{2x} as our third solution. This is how we think to try Cxe^{2x}. **End of Remark**

This is where "the method" comes in: we can quickly write down $P(D)xe^{2x}$ (learning to omit *writing* the in-between steps):

$$P(D)[xe^{2x}] = P(D)\left(\frac{\partial}{\partial r}\right)e^{rx}\bigg|_{r=2} = \left(\frac{\partial}{\partial r}\right)P(D)e^{rx}\bigg|_{r=2} = \left(\frac{\partial}{\partial r}\right)P(r)e^{rx}\bigg|_{r=2} = P'(2)e^{2x} + P(2)xe^{2x} = P'(2)e^{2x}$$

because $P(2) = 0$. Here, $P'(2) = 2 \cdot 2 - 3 = 1$, so we choose $C = 1$ and get $P(D)[xe^{2x}] = e^{2x}$. This means

the general solution is

$$y(x) = y_c(x) + y_p(x) = c_1e^x + c_2e^{2x} + xe^{2x}.$$

All we did was remember the case $k = 1$ of the Leibniz Rule above, because we only duplicated one root.

Example 3 (Two roots duplicated): Find the general solution of the ODE $y'' - 6y' + 9y = e^{3x}$. This time we have $P(r) = (r-3)^2$ so we have a double root of 3, and thus $y_c(x) = c_1e^{3x} + c_2xe^{3x}$. Since it won't work to try $y_p(x) = xe^{3x}$, we try Cx^2e^{3x}. According to Leibniz' Rule, case $k = 2$,

$$P(D)[x^2e^{3x}] = P''(3)e^{3x} + 2P'(3)xe^{3x} + P(3)x^2e^{3x}.$$

Since $P(r) = (r-3)^2$, $P'(r) = 2(r-3)$ and $P''(r) = 2$. When we plug in $r = 3$ we get $P(3) = 0 = P'(3)$ and $P''(3) = 2$. Thus $P(D)[x^2e^{3x}] = 2e^{3x}$. If we take $C = 1/2$ we then have $P(D)[(1/2)x^2e^{3x}] = e^{3x}$, as desired. **The general solution is** $y(x) = y_c(x) + y_p(x) = c_1e^{3x} + c_2xe^{3x} + (1/2)x^2e^{3x}$.

Example 4 (A complex root duplicated): Solve the IVP $y'' + y = 2\cos x$, $y(0) = 1$, $y'(0) = 2$. As a first step we write $\cos x = e^{ix} + e^{-ix}$. Now we can work with $r = i$ and $r = -1$. Our complementary solution is $y_c = c_1e^{ix} + c_2e^{-ix}$ in complex form. But since our linear differential operator has constant real coefficients, if a complex-valued function $y(x)$ satisfies $y'' + y = 0$, then also $\bar{y}'' + \bar{y} = 0$, so $\bar{y}$ is also a solution of the DE. But then $(y + \bar{y})/2$ is a solution, by linearity, and $(y + \bar{y})/2$ is just the real part of y. Similarly, the imaginary part of y, namely $(y - \bar{y})/2i$, is a solution. So if we just don't worry about complex numbers, we can use the method of Example 2, "twice at once." We therefore try $y_p = Axe^{ix} + Bxe^{-ix}$. By our method, we calculate $P(r) = r^2 + 1 = 0$ when $r = \pm i$, and we have $P'(\pm i) = \pm 2i$. Then we use Leibniz:

$$P(D)y_p = AP(D)[xe^{ix}] + BP(D)[xe^{-ix}] = A[P'(i)e^{ix} + P(i)xe^{ix}] + B[P'(-i)e^{-ix} + P(-i)xe^{-ix}].$$

Notice that we don't actually do the differentiations each time! Leibniz' Rule takes care of that! And we treat the complex numbers as just numbers! Now $P(i) = 0 = P(-i)$, so

$$P(D)y_p = AP'(i)e^{ix} + BP'(-i)e^{-ix} = 2i(Ae^{ix} - Be^{-ix}) = 2\cos x$$

if $2iA = 1$ and $-2iB = 1$, by our first step. So we let $A = 1/2i = -i/2$ and we let $B = -1/2i = i/2$. This means that **our particular solution is**

$$y_p = Axe^{ix} + Bxe^{-ix} = (ix/2)[-e^{ix} + e^{-ix}] = (ix/2)[-\cos x - i\sin x + \cos x - i\sin x] = x\sin x.$$

The general solution is $y(x) = C_1\cos x + C_2\sin x + x\sin x$.

Check of this solution (in case you worry about our method):
$y(x) = C_1\cos x + C_2\sin x + x\sin x$.
$y'(x) = -C_1\sin x + C_2\cos x + \sin x + x\cos x$.
$y''(x) = -C_1\cos x - C_2\sin x + \cos x + \cos x - x\sin x$.

Thus
$$\begin{aligned} y''(x) + y(x) &= (-C_1\cos x - C_2\sin x + \cos x + \cos x - x\sin x) + (C_1\cos x + C_2\sin x + x\sin x) \\ &= (2\cos x - x\sin x) + x\sin x = 2\cos x. \end{aligned}$$

Solve the IVP:
$$\begin{aligned} 1 = y(0) &= C_1\cos 0 + C_2\sin 0 + 0\sin 0 = C_1 \\ 2 = y'(0) &= -C_1\sin 0 + C_2\cos 0 + \sin 0 + 0\cos 0 = C_2. \end{aligned}$$

Solution of the IVP: $y(x) = \cos x + 2\sin x + x\sin x$.

Example 5 (Another complex root duplicated, and a power of x on the right):
Solve the IVP $y'' + y = x^2\sin x$, $y(0) = 2$, $y'(0) = 1$. The x^2 on the right means lots more work!

First step: $x^2\sin x = x^2(e^{ix} - e^{-ix})/2i$. Since our $P(D)$ is the same as the one in Example 4, we can use what we did there. We count roots: both of $\pm i$ occur on the left, and both occur on the right. Moreover, there is an x^2, and that counts as another two each of $\pm i$ on the right. The grand total, on the right side is thus three of each of $\pm i$, and each of those occurs once on the left side. We might like to think now that we should try, for a particular solution, $Ax^3e^{ix} + Bx^3e^{-ix}$. But to be safe, we have to try the "in-between" powers of x too:

$$y_t(x) = A_3x^3e^{ix} + A_2x^2e^{ix} + A_1xe^{ix} + B_3x^3e^{-ix} + B_2x^2e^{-ix} + B_1xe^{-ix}.$$

Perhaps we ought to include A_0e^{ix} and B_0e^{-ix}? We don't have to do that, because $P(D)$ annihilates them.

Were we to calculate y_t' and y_t'' directly, without using Leibniz' Rule, we'd have a total of eight terms, with coefficients that are sums, then we'd have to add those, getting eight terms with more complicated coefficients. There would be plenty of room for arithmetic errors. If you are always careful without having to work too hard at it, then you should just keep on. I myself find it safer to use Leibniz' Rule. In what follows, the steps are done "fully," so you can see what's happening. In use, we can skip directly to the step below labelled $(*)$. Here are our formulas, applied first with *any* $P(r)$:

$$\begin{aligned}
P(D)[x^3e^{rx}] &= \left(\frac{\partial}{\partial r}\right)^3 [P(r)e^{rx}] = P'''(r)e^{rx} + 3P''(r)xe^{rx} + 3P'(r)x^2e^{rx} + P(r)x^3e^{rx},\\
P(D)[x^2e^{rx}] &= \left(\frac{\partial}{\partial r}\right)^2 [P(r)e^{rx}] = P''(r)e^{rx} + 2P'(r)xe^{rx} + P(r)x^2e^{rx},\\
P(D)[xe^{rx}] &= \left(\frac{\partial}{\partial r}\right) [P(r)e^{rx}] = P'(r)e^{rx} + P(r)xe^{rx}.
\end{aligned}$$

For our $P(r) = r^2 + 1$ we have $P'(r) = 2r$, $P''(r) = 2$ and $P'''(r) = 0$. This gives

$$\begin{aligned}
P(D)[x^3e^{rx}] &= 6xe^{rx} + 6rx^2e^{rx} + P(r)x^3e^{rx},\\
P(D)[x^2e^{rx}] &= 2e^{rx} + 4rxe^{rx} + P(r)x^2e^{rx},\\
P(D)[xe^{rx}] &= 2re^{rx} + P(r)xe^{rx}.
\end{aligned}$$

When we substitute $r = \pm i$ in, the $P(r)$ terms disappear and we get

$$(*) \qquad \begin{aligned}
P(D)[x^3e^{ix}] &= 6xe^{ix} + 6ix^2e^{ix},\\
P(D)[x^2e^{ix}] &= 2e^{ix} + 4ixe^{ix},\\
P(D)[xe^{ix}] &= 2ie^{ix}
\end{aligned}$$

and

$$\begin{aligned}
P(D)[x^3e^{-ix}] &= 6xe^{-ix} - 6ix^2e^{-ix},\\
P(D)[x^2e^{-ix}] &= 2e^{-ix} - 4ixe^{-ix},\\
P(D)[xe^{-ix}] &= -2ie^{-ix}.
\end{aligned}$$

Now we can try our solution:

$$\begin{aligned}
P(D)[y_t(x)] &= P(D)[A_3x^3e^{ix} + A_2x^2e^{ix} + A_1xe^{ix} + B_3x^3e^{-ix} + B_2x^2e^{-ix} + B_1xe^{-ix}]\\
&= A_3P(D)[x^3e^{ix}] + A_2P(D)[x^2e^{ix}] + A_1P(D)[xe^{ix}]\\
&\quad + B_3P(D)[x^3e^{-ix}] + B_2P(D)[x^2e^{-ix}] + B_1P(D)[xe^{-ix}]\\
&= A_3[6xe^{ix} + 6ix^2e^{ix}] + A_2[2e^{ix} + 4ixe^{ix}] + A_1[2ie^{ix}]\\
&\quad + B_3[6xe^{-ix} - 6ix^2e^{-ix}] + B_2[2e^{-ix} - 4ixe^{-ix}] + B_1[-2ie^{-ix}]\\
&= 6A_3xe^{ix} + 6iA_3x^2e^{ix} + 2A_2e^{ix} + 4iA_2xe^{ix} + 2iA_1e^{ix}\\
&\quad + 6B_3xe^{-ix} - 6iB_3x^2e^{-ix} + 2B_2e^{-ix} - 4iB_2xe^{-ix} - 2iB_1e^{-ix}\\
&= 6iA_3x^2e^{ix} + [6A_3 + 4iA_2]xe^{ix} + [2A_2 + 2iA_1]e^{ix}\\
&\quad - 6iB_3x^2e^{-ix} + [6B_3 - 4iB_2]xe^{-ix} + [2B_2 - 2iB_1]e^{-ix}.
\end{aligned}$$

This explains why we had to include the in-between terms; without them, we'd have "stuff left over;" we need the A_2 to make the coefficient $[6A_3 + 4iA_2] = 0$, and so on. Now we need

$$x^2 \sin x = x^2(e^{ix} - e^{-ix})/2i = 6iA_3x^2e^{ix} - 6iB_3x^2e^{-ix}, \quad \text{or:} \quad 6iA_3 = 1/2i \quad \text{and} \quad 6iB_3 = 1/2i.$$

This means we take $A_3 = -1/12 = B_3$. This gives us

$$\begin{aligned} P(D)[y_t(x)] = {} & (-i/2)x^2e^{ix} + [6A_3 + 4iA_2]xe^{ix} + [2A_2 + 2iA_1]e^{ix} \\ & + (i/2)x^2e^{-ix} + [6B_3 - 4iB_2]xe^{-ix} + [2B_2 - 2iB_1]e^{-ix}. \end{aligned}$$

We want all the other coefficients to be zero. That is, $0 = [6A_3 + 4iA_2] = -1/2 + 4iA_2$ so $A_2 = -i/8$, so $0 = 2A_2 + 2iA_1 = -i/4 + 2iA_1$ so $A_1 = 1/8$. Similarly (please check my work!) $B_2 = i/8$ and $B_1 = 1/8$. This gives

$$P(D)[y_t(x)] = (-i/2)x^2e^{ix} + (i/2)x^2e^{-ix} = x^2(e^{ix} - e^{-ix})/2i = x^2 \sin x,$$

as desired (we have $-i/2 = 1/2i$ and $i/2 = -1/2i$ because $-i = 1/i$).

Our particular solution is $y_p(x) = A_3x^3e^{ix} + A_2x^2e^{ix} + A_1xe^{ix} + B_3x^3e^{-ix} + B_2x^2e^{-ix} + B_1xe^{-ix}$. We can rewrite this by combining the terms with e^{ix} and e^{-ix} that have the same power of x. This gives (putting in the numbers and combining)

$$y_p(x) = -(1/6)x^3 \cos x + (1/4)x^2 \sin x + (1/4)x \cos x.$$

The general solution is $y(x) = C_1 \cos x + C_2 \sin x - (1/6)x^3 \cos x + (1/4)x^2 \sin x + (1/4)x \cos x$.

Solve the IVP:

$$\begin{aligned} 2 &= y(0) = C_1 \\ 1 &= y'(0) = C_2 + (1/4) \quad \text{so} \quad C_2 = 3/4. \end{aligned}$$

Solution of the IVP: $y(x) = 2\cos x + (3/4)\sin x - (1/6)x^3 \cos x + (1/4)x^2 \sin x + (1/4)x \cos x$.

Closing Remarks

Using this method requires some practice, and you may well decide not to use it. If so, be sure to work along with the examples given in Sections 2.4 and 2.5. The presentation here makes it seem long, but with practice you learn which steps you can combine or even skip.