§1: Un-ordered Summation: non-negative numbers

We begin with a non-empty set A, and a function $a\colon A \to \mathbb{R}$, such that, for all $\alpha \in A$, $a_\alpha \geq 0$. Our most-used example will be $A = \mathbb{N} \times \mathbb{N}$, and then, a_α will be written a_{mn}, not $a_{(m,n)}$.

Suppose that F is a *finite* subset of A. We will write $\displaystyle\sum_{\alpha\in F} a_\alpha$ to denote the sum of all the numbers a_α such that $\alpha \in F$. We *could* specify a way to express this sum more explicitly, but it would not make any difference in the sum because addition is commutative and associative. I emphasize that here F is a *finite* set!

Now we can *define* $\displaystyle\sum_{\alpha\in A} a_\alpha$ to be the supremum of the **set** of *all* sums $\displaystyle\sum_{\alpha\in F} a_\alpha$, where F is a *finite* subset of A.

In symbols, $$\sum_{\alpha\in A} a_\alpha := \sup\{y \in \mathbb{R} : \exists F \subseteq A \text{ such that } F \text{ is finite, and } y = \sum_{\alpha\in F} a_\alpha\,\} = \sup \sum_{\alpha\in F} a_\alpha,$$

where the supremum at the end means the least upper bound as F "ranges" over *all* the finite subsets of A. It is a very informal notation that does a good job of getting the idea across, but can be confusing if pressed too hard to convey information completely and precisely. We could achieve a compact presentation by first defining a set

$$(1.1)\qquad \mathcal{S} = \mathcal{S}(A) := \{y \in \mathbb{R} : \exists F \subseteq A \text{ such that } F \text{ is finite, and } y = \sum_{\alpha\in F} a_\alpha\,\}, \quad \text{and then write} \quad \sum_{\alpha\in A} a_\alpha := \sup \mathcal{S}.$$

Please Note: Nothing was said so far about the set $\mathcal{S}(A)$ being bounded above! If the set of finite sums, $\mathcal{S}$, is nonempty and NOT bounded above, we say that "the un-ordered sum is infinite," and we write $\displaystyle\sum_{\alpha\in A} a_\alpha = +\infty$. We always do this – say that a set that is nonempty and not bounded above has least upper bound equal to $+\infty$. This may explain why we simply say, as above, that $\displaystyle\sum_{\alpha\in A} a_\alpha = \sup \mathcal{S}$. Aside (important!): what if S is empty?

We can state a useful and natural lemma now, that we will use later without mention. Its proof is trivial.

(1.2) **Lemma:** *Let A be a non-empty set, and let a and b be functions $a\colon A \to \mathbb{R}$ and $b\colon A \to \mathbb{R}$ such that, for all $\alpha \in A$, $0 \leq a_\alpha \leq b_\alpha$. Then* $\displaystyle\sum_{\alpha\in A} a_\alpha \leq \sum_{\alpha\in A} b_\alpha$.

The main tool for working with unordered sums is the "big enough" partial sum:

(1.3) **Approximation Theorem (part 1):** *Let A be a non-empty set, and let a be a function $a\colon A \to \mathbb{R}$ such that, for all $\alpha \in A$, $a_\alpha \geq 0$. If $\displaystyle\sum_{\alpha\in A} a_\alpha < +\infty$, then for all $\epsilon > 0$, there exists a finite set $F \subseteq A$ such that*

$$\textit{for all finite sets } G \subseteq A, \sum_{\alpha\in G\sim F} a_\alpha < \epsilon.$$

In other words, those numbers a_α, with α in a finite set G, whose subscripts α *don't* belong to F, add up to a total that is less than ϵ. In case $G \sim F = \emptyset$, we have an empty sum, which we define to be 0.

Proof : We are given that the set $\mathcal{S} = \mathcal{S}(A)$, defined in (1) is bounded above. Thus, for all $\epsilon > 0$, there exists $y \in \mathcal{S}$ such that $y > \sup \mathcal{S} - \epsilon$. Since $y \in \mathcal{S}$, there exists a finite set F such that $y = \displaystyle\sum_{\alpha\in F} a_\alpha$. Therefore,

$$\sum_{\alpha\in F} a_\alpha = y > \sup \mathcal{S} - \epsilon = \sum_{\alpha\in A} a_\alpha - \epsilon.$$

Reminder: The symbols $\sum_{\alpha\in F} a_\alpha$ and $\sum_{\alpha\in A} a_\alpha$ are defined differently!

Now, let G be any finite subset of A. Then

$$\sum_{\alpha\in A} a_\alpha \geq \sum_{\alpha\in G\cup F} a_\alpha = \sum_{\alpha\in G\sim F} a_\alpha + \sum_{\alpha\in F} a_\alpha > \sum_{\alpha\in G\sim F} a_\alpha + \sum_{\alpha\in A} a_\alpha - \epsilon. \tag{1.4}$$

By looking at the two ends of the relations in (1.4) we can see that the proof is done.

We can give a version of the Approximation Theorem for unordered sums that are not finite:

(1.5) **Approximation Theorem (part 2):** *Let A be a non-empty set, and let a be a function $a\colon A \to \mathbb{R}$ such that, for all $\alpha \in A$, $a_\alpha \geq 0$.*
If $\sum_{\alpha\in A} a_\alpha = +\infty$, then for all real R, there exists a finite set $F \subseteq A$ such that for all finite sets $G \supseteq F$, $\sum_{\alpha\in G} a_\alpha > R$.

Exercise: Prove this.

We need one more lemma. But first you may need to work this
Exercise: *Suppose that $\mathcal{S}$ and $\mathcal{T}$ are nonempty sets of real numbers. We define*

$$\mathcal{S}+\mathcal{T} := \{x \in \mathbb{R} : x = s + t \text{ such that } s \in \mathcal{S} \text{ and } t \in \mathcal{T}\}.$$

Prove that $\sup(\mathcal{S}+\mathcal{T}) = \sup\mathcal{S} + \sup\mathcal{T}$. Don't forget that we have *not* assumed the sets are bounded! For example, $(-1, 1) + [0, \infty) = (-1, \infty)$. It is useful to think of the "sum" of two sets as the union of all the "shifts" of one of the sets by elements of the other set.

(1.6) **Lemma:** *Let A, B and C be nonempty sets such that $A \cup B = C$ and $A \cap B = \emptyset$. Let a and b be functions $a\colon A \to \mathbb{R}$ and $b\colon B \to \mathbb{R}$ such that for all $\alpha \in A$, $a_\alpha \geq 0$ and for all $\beta \in B$, $b_\alpha \geq 0$. Let $c\colon C \to \mathbb{R}$ be given by $c_\gamma := \begin{cases} a_\gamma, & \text{if } \gamma \in A, \\ b_\gamma, & \text{if } \gamma \in B. \end{cases}$ Then $\sum_{\alpha\in A} a_\alpha + \sum_{\beta\in B} b_\beta = \sum_{\gamma\in C} c_\gamma$.*

Proof : Let $F \subseteq C$ be finite and nonempty. Then $F \cap A \subseteq A$ is finite, and $F \cap B \subseteq B$ is finite. Therefore

$$\sum_{\gamma\in F} c_\gamma = \sum_{\gamma\in F\cap A} c_\gamma + \sum_{\gamma\in F\cap B} c_\gamma = \sum_{\gamma\in F\cap A} a_\gamma + \sum_{\gamma\in F\cap B} b_\gamma \leq \sum_{\alpha\in A} a_\alpha + \sum_{\beta\in B} b_\beta,$$

by definition. But on the left-hand side, F was an arbitrary finite set, so

$$\sum_{\gamma\in C} c_\gamma := \sup_{F\subseteq C,\ F \text{ finite}} \sum_{\gamma\in F} c_\gamma \leq \sum_{\alpha\in A} a_\alpha + \sum_{\beta\in B} b_\beta.$$

Given $R < \sum_{\alpha\in A} a_\alpha$ and $S < \sum_{\beta\in B} b_\beta$ there exist *finite* sets $F \subseteq A$ and $G \subseteq B$ such that

$$\sum_{\alpha\in F} a_\alpha > R \quad\text{and}\quad \sum_{\beta\in G} b_\beta > S, \quad\text{so that}\quad \sum_{\alpha\in F} a_\alpha + \sum_{\beta\in G} b_\beta > R + S.$$

But then

$$\sum_{\gamma\in C} c_\gamma \geq \sum_{\gamma\in F\cup G} c_\gamma = \sum_{\alpha\in F} a_\alpha + \sum_{\beta\in G} b_\beta > R + S.$$

Since $\sum_{\alpha\in A} a_\alpha + \sum_{\beta\in B} b_\beta$ is the least upper bound of all the numbers $R + S$ that we used, the conclusion of the Lemma follows from the fact that when $\mathcal{S}$ and $\mathcal{T}$ are nonempty sets of real numbers, $\sup(\mathcal{S}+\mathcal{T}) = \sup\mathcal{S} + \sup\mathcal{T}$.
Remark: The $R + S$ part of the argument allows us to deal with the finite-sum and infinite-sum cases in one argument, so it needs to allow for "sups" of unbounded sets.

Another way to say this: if C and D are disjoint sets, (disjoint means $C \cap D = \emptyset$) then $\sum_{\alpha\in C\cup D} a_\alpha = \sum_{\alpha\in C} a_\alpha + \sum_{\alpha\in D} a_\alpha$.
This can be extended to any finite number of subsets of A that are pairwise disjoint. Recall that we write $\sum_{\alpha\in G} a_\alpha = 0$, if $G = \emptyset$.

(1.7) **Theorem(at most countably many positive terms):** *Let* A *be a non-empty set, and let* a *be a function* $a\colon A \to \mathbb{R}$ *such that, for all* $\alpha \in A$, $a_\alpha \ge 0$.

$$\text{If } \sum_{\alpha \in A} a_\alpha < +\infty, \text{ then the set } P := \{\alpha \in A : a_\alpha > 0\} \text{ is at most countable.}$$

In other words, the *positive* terms can be listed in a sequence if the un-ordered sum is finite.

Proof : We will divide the set P into pieces, and show that each piece is a finite set. Then, we'll be able to express P as a countable union of at most countable sets, which will show that P is at most countable.

For each natural number N, let D_N denote the set of all $\alpha \in A$ such that $\dfrac{1}{2^{N+1}} < a_\alpha \le \dfrac{1}{2^N}$. Let D_0 denote the set of all $\alpha \in A$ such that $1 < a_\alpha$. If $0 < a_\alpha \le 1$, there is a largest N such that $a_\alpha \le \dfrac{1}{2^N}$, so for this N, $\dfrac{1}{2^{N+1}} < a_\alpha$, so $\alpha \in D_N$. It follows easily that P is the union of D_0 and all the sets D_N.

Is D_0 a finite set? To answer this we let $S := \sum_{\alpha \in A} a_\alpha$, and define $M := [S]$, the greatest integer in S.

Let's show that D_0 has at most M elements. Otherwise, there would exist distinct elements $\alpha_1, \alpha_2, \ldots, \alpha_{M+1}$ in D_0. But then, if we set $G := \{\alpha_1, \alpha_2, \ldots, \alpha_{M+1}\}$,

$$M + 1 > S \ge \sum_{\alpha \in G} a_\alpha = \sum_{i=1}^{M+1} a_{\alpha_i} > \sum_{i=1}^{M+1} 1 = M + 1,$$

a contradiction. Thus, D_0 has at most $[S]$ elements. In a similar way, you can show that D_N has at most $[2^{N+1}S]$ elements. Incidentally, we did not have to define the D_N's so they were disjoint! Usually, this argument is given with D_N defined to be the set of all α such that $a_\alpha > 1/2^N$, because we never used the upper bound on the sizes of the a_α! At any rate, the proof is finished by using the fact that a countable union of at most countable sets is an at most countable set.

Now that we know the set P of positive a_α is at most countable if $\sum_{\alpha \in A} a_\alpha$ is finite, we know there exists a one-to-one correspondence between P and, either $\mathbb{N}$ if P *is* countable, or $S_N := \{n \in \mathbb{N} : 0 \le n < N\}$, where $N \in \mathbb{N}$, that associates each natural number n (in $\mathbb{N}$ or S_N, as appropriate), with one and only one $\alpha \in P$, denoted α_n. Such a one-to-one correspondence is called an **enumeration** of a set. In this case it is the set P that is **enumerated**, either by $\mathbb{N}$ or by S_N. Here, it is useful to make $\mathbb{N}$ start with zero! *Why?*

Aside (unimportant): There are either uncountably many such enumerations of P or there are $N!$ of them.

(1.8) **Theorem(on expressing an un-ordered sum as a series):** *Let* $\sum_{\alpha \in A} a_\alpha$ *be a finite and positive un-ordered sum of non-negative real numbers. If the set* P *of all* $\alpha \in A$ *such that* $a_\alpha > 0$ *is enumerated as* $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots,$ *(where the enumeration may terminate with some finite* N_o*), then* $\sum_{\alpha \in A} a_\alpha = \sum_{n=0}^{\infty} a_{\alpha_n}$.

Proof : Without loss of generality, we may assume that P is countable. Let $\epsilon > 0$ be given. There exists a finite set F such that $\sum_{\alpha \in F} a_\alpha > \sum_{\alpha \in A} a_\alpha - \epsilon$. Since $F \cap P$ is a finite set, there exists a largest n, say N, such that $\alpha_n \in P$. That is, for all natural numbers n, if $\alpha_n \in F$, then $n \le N$. Thus $F \subseteq \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_N\}$, so

$$0 \le \sum_{\alpha \in A} a_\alpha - \sum_{\alpha \in \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_N\}} a_\alpha \le \sum_{\alpha \in A} a_\alpha - \sum_{\alpha \in F} a_\alpha < \epsilon.$$

Therefore,

$$0 \le \sum_{\alpha \in A} a_\alpha - \sum_{\alpha \in \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_N\}} a_\alpha = \sum_{\alpha \in A} a_\alpha - \sum_{n=0}^{N} a_{\alpha_n} < \epsilon.$$

For all $M \geq N$, we thus have $\displaystyle\sum_{\alpha\in A} a_\alpha - \sum_{n=0}^{M} a_{\alpha_n} < \epsilon$. Because ϵ was arbitrary, we have shown that the sequence of partial sums of the series $\displaystyle\sum_{n=0}^{\infty} a_{\alpha_n}$ converges to $\displaystyle\sum_{\alpha\in A} a_\alpha$, which is what was to be shown.

A Problem

Show that, if $\{a_n\}$ *is a sequence of non-negative numbers, then* $\displaystyle\sum_{n\in\mathbb{N}} a_n = \sum_{n=0}^{\infty} a_n$. *Do not use the theorem on rearrangements of absolutely convergent series that you learned in Calculus!* Recall: the sum on the left is un-ordered!

The purpose of this Problem is to show you that the proof of the preceding Theorem, plus this Problem, essentially proves the theorem on rearrangements of absolutely convergent series! Your proof should include the case of series that diverge to $+\infty$. The proof of the next Theorem might help you to solve the problem, but you really only need to use facts about the sequence of partial sums of a series with non-negative terms, and the *definition* of an unordered sum.

A Corollary of the Problem: *Suppose that* $\{a_n\}$ *is a sequence of non-negative numbers, and that* $\displaystyle\sum_{n=0}^{\infty} a_n < \infty$. *Then, for all rearrangements* $\{n_j\}_{j=0}^{\infty}$ *of the non-negative integers,* $\sum_{j=0}^{\infty} a_{n_j} = \sum_{n=0}^{\infty} a_n$.

Please note that $\{n_j\}_{j=0}^{\infty}$ is not a subsequence; it is actually a one-to-one correspondence between $\mathbb{N}$, indexed by j, and $\mathbb{N}$, indexed by n. An example: $\{0, 2, 1, 3, 5, 4, 6, 8, 10, \ldots\}$, two evens, three odds, next 4 evens, etc.

A useful feature of a rearrangement of $\mathbb{N}$ is this: for every finite subset S of $\mathbb{N}$, there exists $J \in \mathbb{N}$ such that S is contained in $\{n_0, n_1, \ldots n_J\}$.

However, there are ways of "rearranging" $\mathbb{N}$ that don't "recapture" all of $\mathbb{N}$ in one "pass" thru $\mathbb{N}$. For example, we might make all the evens, in natural order, come before the odds, also in natural order. This ordering is *not* a rearrangement of $\mathbb{N}$ in the sense just discussed!

The next Theorem is what allows us to deal with "rearrangements" that are not "permutations," and with multiple series of non-negative numbers. As stated, it leads directly to double series. But it can be "iterated" to handle series with any finite number of subscripts. We will generalize the Theorem later.

(1.9) **Theorem (on expressing an un-ordered sum as a series of un-ordered "sub-sums"):**
Let $\displaystyle\sum_{\alpha\in A} a_\alpha$ *be an un-ordered sum of non-negative real numbers.*

If the index set A *is the disjoint union of sets* A_n, *that is, if* $A = \displaystyle\bigcup_{n=0}^{\infty} A_n$, *where* $A_n \cap A_m = \emptyset$ *if* $n \neq m$,

then $\displaystyle\sum_{\alpha\in A} a_\alpha = \sum_{n=0}^{\infty}\left(\sum_{\alpha\in A_n} a_\alpha\right)$.

For example, if $A = \mathbb{N}\times\mathbb{N}$, so that a_α is written a_{mn}, not $a_{(m,n)}$, and $A_n := \{(m, n) : m = 0, 1, \ldots\}$ for $n = 0, 1, \ldots$, the theorem says we can add columnwise first, and add those sums:

$$\sum_{\mathbb{N}\times\mathbb{N}} a_{mn} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{mn}\right).$$

Recall that, if $S = \emptyset$, then we define $\displaystyle\sum_{\alpha\in S} a_\alpha = 0$. Let us look at some examples before proving the Theorem.

Example 1: $A = \mathbb{Z}^+$, A_0 is the set of even positive integers, A_1 is the set of odd positive integers, and $A_n = \emptyset$, if $n > 1$.

The Theorem "says," after simplification, that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty}\frac{1}{(2n)^2}+\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$. We can then write

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty}\frac{1}{(2n)^2}+\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2},\quad\text{so that}\quad\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Example 2: $A=\mathbb{Z}^+$, $A_n:=\{2^n(2k-1):k\in\mathbb{Z}^+\}$, so $A_n=2^nA_0$, for all natural numbers n.

The Theorem "says," after simplification, that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(2^n(2k-1))^2}$. We can then write

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=0}^{\infty}\frac{1}{2^{2n}}\times\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{4}{3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

Proof of Theorem(1.9): **The finite cases.**

First we suppose that $\sum_{\alpha\in A}a_\alpha$ is finite. Then, since $A_n\subseteq A$ for all $n\in\mathbb{N}$, $\sum_{\alpha\in A_n}a_\alpha\le\sum_{\alpha\in A}a_\alpha$ for all n. Indeed, we can apply Lemma (1.6) and induction to conclude that for any positive N, $\sum_{n=1}^{N}\sum_{\alpha\in A_n}a_\alpha\le\sum_{\alpha\in A}a_\alpha$. But then $\sum_{n=1}^{\infty}\sum_{\alpha\in A_n}a_\alpha\le\sum_{\alpha\in A}a_\alpha$. Let $\epsilon>0$ be given. Define $\epsilon_n:=\epsilon/2^{n+3}$. Then, for all $n\in\mathbb{N}$, there exists a finite subset

$$F_n\subseteq A_n,\quad\text{such that}\quad\sum_{\alpha\in F_n}a_\alpha>\sum_{\alpha\in A_n}a_\alpha-\epsilon_n.$$

There also exists a finite set Φ such that

$$\sum_{\alpha\in\Phi}a_\alpha>\sum_{\alpha\in A}a_\alpha-\epsilon/4.$$

Let $\Psi_n:=F_n\cup(\Phi\cap A_n)\subseteq A_n$. For all $n\in\mathbb{N}$, Ψ_n is the union of two finite sets, hence is finite also. We need Ψ_n for "bookkeeping" purposes. Each Ψ_n is a subset of A_n, and it consists of all the elements of F_n, plus any elements of Φ that are in A_n and not already in F_n.

Since Φ is finite, there is a largest n, say N, such that A_n contains an element of Φ. That is, A_N contains an element of Φ but if $n>N$ then A_n contains *no* elements of Φ. Let $\Psi:=\bigcup_{n=0}^{N}\Psi_n$. Note that $\Phi\subseteq\Psi$. *Why?* Then

$$\sum_{\alpha\in A}a_\alpha-\sum_{n=0}^{\infty}\left(\sum_{\alpha\in A_n}a_\alpha\right)=\sum_{\alpha\in A}a_\alpha-\sum_{\alpha\in\Phi}a_\alpha+\sum_{\alpha\in\Phi}a_\alpha-\sum_{n=0}^{N}\sum_{\alpha\in\Psi_n}a_\alpha+\sum_{n=0}^{N}\sum_{\alpha\in\Psi_n}a_\alpha-\sum_{n=0}^{N}\sum_{\alpha\in A_n}a_\alpha-\sum_{n=N+1}^{\infty}\sum_{\alpha\in A_n}a_\alpha.$$

Therefore, by the way Φ was chosen, the first difference on the right is non-negative and less than $\epsilon/4$, so

$$(1.10)\quad\left|\sum_{\alpha\in A}a_\alpha-\sum_{n=0}^{\infty}\left(\sum_{\alpha\in A_n}a_\alpha\right)\right|<\epsilon/4+\left|\sum_{\alpha\in\Phi}a_\alpha-\sum_{n=0}^{N}\sum_{\alpha\in\Psi_n}a_\alpha\right|+\left|\sum_{n=0}^{N}\sum_{\alpha\in\Psi_n}a_\alpha-\sum_{n=0}^{N}\sum_{\alpha\in A_n}a_\alpha\right|+\sum_{n=N+1}^{\infty}\sum_{\alpha\in A_n}a_\alpha.$$

Let's show that the last term is less than $\epsilon/4+\epsilon/2^{N+3}$. For each $n>N$, we have chosen a finite set F_n such that $\sum_{\alpha\in A_n}a_\alpha<\sum_{\alpha\in F_n}a_\alpha+\epsilon/2^{n+3}$. Therefore,

$$\sum_{n=N+1}^{\infty}\sum_{\alpha\in A_n}a_\alpha<\sum_{n=N+1}^{\infty}\left(\sum_{\alpha\in F_n}a_\alpha+\epsilon/2^{n+3}\right)=\left(\sum_{n=N+1}^{\infty}\sum_{\alpha\in F_n}a_\alpha\right)+\epsilon/2^{N+3}.$$

Now for each positive integer K, $\displaystyle\sum_{n=N+1}^{N+K}\sum_{\alpha\in F_n} a_\alpha = \sum_{\alpha\in G_K} a_\alpha$, where G_K is the union of the mutually disjoint sets F_n, as n runs through the values $N+1$ to $N+K$, inclusive; G_K is a finite set. Moreover, G_K and Φ are disjoint. By the Approximation Theorem, $\displaystyle\sum_{\alpha\in G_K} a_\alpha < \epsilon/4$. Thus the last term is less than $\epsilon/4+\epsilon/2^{N+3}$.

The first middle term is actually equal to

$$\sum_{n=0}^{N}\sum_{\alpha\in\Psi_n} a_\alpha - \sum_{\alpha\in\Phi} a_\alpha = \sum_{\alpha\in\Psi} a_\alpha - \sum_{\alpha\in\Phi} a_\alpha = \sum_{\alpha\in\Psi\sim\Phi} a_\alpha.$$

Then, since $\Psi\sim\Phi$ is a finite set that is disjoint from Φ, we can once again apply the approximation theorem to see that the first middle term is less than $\epsilon/4$.

So far, we have shown that the sum of all the terms in (3), except the second middle one, is less than $3\epsilon/4$. The second middle term, which we can call SMT, for short, is equal to $\displaystyle SMT := \sum_{n=0}^{N}\left(\sum_{\alpha\in A_n} a_\alpha - \sum_{\alpha\in\Psi_n} a_\alpha\right)$. But Ψ_n contains F_n, so $\displaystyle\sum_{\alpha\in A_n} a_\alpha - \sum_{\alpha\in\Psi_n} a_\alpha \le \sum_{\alpha\in A_n} a_\alpha - \sum_{\alpha\in F_n} a_\alpha < \epsilon/2^{n+3}$. Thus, by a calculation involving the formula for the partial sum of a geometric series, $SMT < \sum_{n=0}^{N}\epsilon/2^{n+3} = \epsilon/4 - \epsilon/2^{N+3}$. This gives us a total estimate of ϵ for the right-hand-side of (3), as desired – ϵ was arbitrary, so the desired equality of the Theorem is true, if the unordered sum is finite. We now need to show that, if the double sum on the right is finite, so is the unordered sum, that was on the left. However, we can deduce this from the infinite cases, so let us deal with those now.

The infinite cases.

Next, we need to consider the two possible "infinite" cases: when the quantity on either side of the asserted equation $\displaystyle\sum_{\alpha\in A} a_\alpha = \sum_{n=0}^{\infty}\left(\sum_{\alpha\in A_n} a_\alpha\right)$ is infinite. We will show that if either quantity is infinite, so is the other. This will complete the finite cases! The arguments are similar to what we have done before, but are actually shorter.

First, suppose the quantity on the left, $\displaystyle\sum_{\alpha\in A} a_\alpha$, is infinite. By definition, this means that the set $\mathcal{S}$ defined in (1.1) is not bounded above, so that, for all large real R, there exists a finite set $F\subseteq A$ such that $\sum_{\alpha\in F} a_\alpha > R$. Because F is finite, there exists a largest n, say N, such that A_n contains any elements of F. Since the sets A_n are disjoint, so are the sets $F_n := A_n\cap F$. Hence, $\displaystyle R < \sum_{\alpha\in F} a_\alpha = \sum_{n=0}^{N}\sum_{\alpha\in F_n} a_\alpha$. Since F_n is finite, $\displaystyle\sum_{\alpha\in F_n} a_\alpha \le \sum_{\alpha\in A_n} a_\alpha$ for $0\le n\le N$. Therefore,

$$R < \sum_{\alpha\in F} a_\alpha = \sum_{n=0}^{N}\sum_{\alpha\in F_n} a_\alpha \le \sum_{n=0}^{N}\sum_{\alpha\in A_n} a_\alpha \le \sum_{n=0}^{\infty}\sum_{\alpha\in A_n} a_\alpha.$$

But R was arbitrary; the only *extended* real number larger than an *arbitrary* real number is $+\infty$; we have thus shown that, if $\displaystyle\sum_{\alpha\in F} a_\alpha$ is infinite, so is $\displaystyle\sum_{n=0}^{\infty}\sum_{\alpha\in A_n} a_\alpha$.

We finally have to show that, if $\displaystyle\sum_{n=0}^{\infty}\sum_{\alpha\in A_n} a_\alpha$ is infinite, so is $\displaystyle\sum_{\alpha\in F} a_\alpha$. Thus, suppose that $\displaystyle\sum_{n=0}^{\infty}\sum_{\alpha\in A_n} a_\alpha$ is infinite. Therefore, for all large real R, there exists a natural number N so large that $\displaystyle R < \sum_{n=0}^{N}\sum_{\alpha\in A_n} a_\alpha$. It could happen that one of the "numbers" $\sum_{\alpha\in A_n} a_\alpha$ is infinite. But then, since $\mathcal{S}$ for $\sum_{\alpha\in A_n} a_\alpha$ is contained in $\mathcal{S}$ for $\sum_{\alpha\in A} a_\alpha$, we would immediately know that $\sum_{\alpha\in A} a_\alpha$ is infinite. So, we have to assume that all the numbers $\sum_{\alpha\in A_n} a_\alpha$ are finite. In that case, we let $\epsilon := \sum_{n=0}^{N}\sum_{\alpha\in A_n} a_\alpha - R$, and we know that there are finite sets $F_n \subseteq A_n$ such that $\sum_{\alpha\in F_n} a_\alpha > \sum_{\alpha\in A_n} a_\alpha - \epsilon/(N+1)$, for $0 \le n \le N$. Let $G := \bigcup_{n=0}^{N} F_n$. Then G is a finite set, the sets $F_n s$ are disjoint, and so

$$\sum_{\alpha\in G} a_\alpha = \sum_{n=0}^{N}\sum_{\alpha\in F_n} a_\alpha > \sum_{n=0}^{N}\left(\sum_{\alpha\in A_n} a_\alpha - \epsilon/(N+1)\right) = \sum_{n=0}^{N}\sum_{\alpha\in A_n} a_\alpha - \sum_{n=0}^{N}\epsilon/(N+1) = \sum_{n=0}^{N}\sum_{\alpha\in A_n} a_\alpha - \epsilon.$$

That is,

$$\sum_{\alpha\in G} a_\alpha > \sum_{n=0}^{N}\sum_{\alpha\in A_n} a_\alpha - \epsilon = \sum_{n=0}^{N}\sum_{\alpha\in A_n} a_\alpha - \left(\sum_{n=0}^{N}\sum_{\alpha\in A_n} a_\alpha - R\right) = R,$$

by the way ϵ was defined. Therefore, $\mathcal{S}$ for $\sum_{\alpha\in A} a_\alpha$ contains arbitrarily large real numbers, and we have shown that $\sum_{\alpha\in A} a_\alpha = +\infty$ if $\sum_{n=0}^{\infty}\sum_{\alpha\in A_n} a_\alpha$ is infinite. The proof of Theorem (1.9) is finally complete.

Here are some **applications** of Theorem (1.9).

Let's investigate double series whose terms have the form $a_{mn} = a_m b_n$, where the a_m and b_n are all non-negative. This theorem will tell us that, *in some circumstances*, we need to define $0 \times (+\infty) = 0$.

Theorem (on series of products)

If $\{a_m\}$ *and* $\{b_n\}$ *are sequences of non-negative real numbers, then* $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_m b_n = \sum_{m=0}^{\infty} a_m \times \sum_{n=0}^{\infty} b_n$, *provided that, on the right-hand side, we interpret* $0 \times (+\infty)$ *or* $(+\infty)\times 0$ *as* 0.

Proof : Let us deal with the "zero times infinity" cases first. If $\sum_{m=0}^{\infty} a_m = 0$, then for all $m \in \mathbb{N}$, $a_m = 0$, so the terms $a_m b_n$ are all equal to zero, and so the set $\mathcal{S}$ for $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_m b_n$ is $\{0\}$, and so $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_m b_n = 0$. The argument is entirely similar if $\sum_{n=0}^{\infty} b_n = 0$.

Next, let's suppose that both of $\sum_{m=0}^{\infty} a_m$ and $\sum_{n=0}^{\infty} b_n$ are positive, and at least one of them is infinite. This means that there is at least one m and at least one n such that $a_m b_n > 0$. Suppose it's $\sum_{n=0}^{\infty} b_n$ that is infinite. Then for some m such that $a_m > 0$, we define the finite sets $F_N := \{(m,n) : 0 \le n \le N\}$, one for each natural number N. We have $\sum_{(\mu,\nu)\in F_N} a_\mu b_\nu = a_m \sum_{0\le n\le N} b_n = a_m \sum_{n=0}^{N} b_n \to +\infty$. Therefore, $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_m b_n = +\infty$, and the product $\sum_{m=0}^{\infty} a_m \times \sum_{n=0}^{\infty} b_n$ is infinite as well.

Please note that we usually don't show that two infinite quantities are equal to each other; we show that each is equal to the same infinity.

Let's apply this to an important specific example.

Theorem (that says $e^{x+y} = e^x e^y$ **)**
for all real numbers x *and* y,

$$\sum_{n=0}^{\infty}\frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{n=0}^{\infty}\frac{y^n}{n!}.$$

Proof : At this time, we won't prove quite what the Theorem says; we'll assume that x and y are both positive. First, please use the Ratio Test to show that all three series in the statement of the Theorem are convergent.

We rewrite the series on the left in the form $\sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{1}{n!} \binom{n}{r} x^{n-r} y^r$. This is valid, because we are only replacing $\frac{(x+y)^n}{n!}$ by $\sum_{r=0}^{n} \frac{1}{n!} \binom{n}{r} x^{n-r} y^r$, for each n. We algebraically simplify $\frac{1}{n!} \binom{n}{r}$ to $\frac{1}{(n-r)!} \frac{1}{r!}$. We then have $\sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{x^{n-r}}{(n-r)!} \frac{y^r}{r!} = \sum_{n=0}^{\infty} \sum_{r+s=n} \frac{x^s}{s!} \frac{y^r}{r!} = \sum_{(s,r) \in \mathbb{N} \times \mathbb{N}} \frac{x^s}{s!} \frac{y^r}{r!} = \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{x^s}{s!} \frac{y^r}{r!}$. This completes the proof, when x and y are non-negative. We will deal with the other cases of the signs of x and y at the end of the next Section. By the way, the sets A_n are $A_n = \{(s,r) \in \mathbb{N} \times \mathbb{N} : r+s=n\}$.

The next example uses the same sets A_n. The problem is: *For which* p *is the sum* $\sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{1}{(s+r+1)^p}$ *finite?*

We can write $\sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \frac{1}{(s+r+1)^p} = \sum_{n=0}^{\infty} \sum_{r+s=n} \frac{1}{(s+r+1)^p} = \sum_{n=0}^{\infty} \sum_{r+s=n} \frac{1}{(n+1)^p} = \sum_{n=0}^{\infty} \frac{(n+1)}{(n+1)^p}$, and we know this is finite if and only if $p > 2$.

§2: Un-ordered Summation: real numbers

We can use the same arguments that we used before even if the numbers a_α can be either positive, negative, or zero. The only problem we have is when the sum of the positive parts of the a_α and the negative parts of the a_α both have infinite sums. We simply can't tell what value to assign to $\infty - \infty$! But if $\sum_{\alpha \in A} a_\alpha^+ < \infty$, then, whether $\sum_{\alpha \in A} a_\alpha^-$ is finite or not, their *difference* $\sum_{\alpha \in A} a_\alpha^+ - \sum_{\alpha \in A} a_\alpha^-$ has a definite meaning. Thus we make the following definition:

(2.1) **Definition(of un-ordered sums of real numbers):** *We begin with a non-empty set* A, *and a function* $a\colon A \to \mathbb{R}$. *Then, if at least one of the numbers*

$\sum_{\alpha \in A} a_\alpha^+$ *and* $\sum_{\alpha \in A} a_\alpha^-$ *is finite, we define* $\sum_{\alpha \in A} a_\alpha := \sum_{\alpha \in A} a_\alpha^+ - \sum_{\alpha \in A} a_\alpha^-$.

If both of $\sum_{\alpha \in A} a_\alpha^+$ *and* $\sum_{\alpha \in A} a_\alpha^-$ *are infinite, we say that* $\sum_{\alpha \in A} a_\alpha$ *is undefined.*

The point is, in the theory of un-ordered sums there is no such thing as a conditionally convergent series. Conditionally convergent series arise in connection with an *ordering* of partial sums. The theory of un-ordered sums is concerned only with sums defined in terms of least upper bounds.

There are some straightforward theorems. **Exercise:** *Prove them.*

(2.2) $\sum_{\alpha \in A} a_\alpha$ *is finite if and only if* $\sum_{\alpha \in A} |a_\alpha|$ *is finite.*

(2.3) *If* $\sum_{\alpha \in A} a_\alpha$ *is defined, finite or not, and* $\pi\colon A \to A$ *is one-to-one and onto, then* $\sum_{\alpha \in A} a_{\pi\alpha} = \sum_{\alpha \in A} a_\alpha$. *Moreover, one of these sums is defined if and only if the other one is.*

(2.4) *If* $\sum_{\alpha \in A} a_\alpha$ *is finite, then at most denumerably many of the terms* a_α *are non-zero.*

(2.5) *If* $\sum_{\alpha \in A} a_\alpha$ *is finite, then for all* $\epsilon > 0$, *there exists a finite set* $F \subseteq A$ *such that,*

for all finite sets $G \subseteq A$, $\sum_{\alpha \in G \sim F} |a_\alpha| < \epsilon$.

(2.6) *Suppose* $\sum_{\alpha\in A} a_\alpha$ *is finite.*
If the set P, *of all* $\alpha \in A$ *such that* $a_\alpha \neq 0$, *is enumerated as* $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots,$ *(where the enumeration may terminate with some finite* N_o), *then* $\sum_{\alpha\in A} a_\alpha = \sum_{n=0}^{\infty} a_{\alpha_n}$.

(2.7) *If* $\sum_{\alpha\in A} a_\alpha$ *is defined, and if the index set* A *is the disjoint union of sets* A_n, *that is, if* $A = \bigcup_{n=0}^{\infty} A_n$, *where* $A_n \cap A_m = \emptyset$ *if* $n \neq m$, *then* $\sum_{\alpha\in A} a_\alpha = \sum_{n=0}^{\infty}\left(\sum_{\alpha\in A_n} a_\alpha\right)$.

Now we return to the exponential series. For all real numbers x and y,

$$\sum_{n=0}^{\infty}\frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{n=0}^{\infty}\frac{y^n}{n!}.$$

All we have to do is apply (2.2) and (2.7). We already know that if $x \neq 0$ and $y \neq 0$ then

$$\sum_{n=0}^{\infty}\frac{|x+y|^n}{n!} \leq \sum_{n=0}^{\infty}\frac{(|x|+|y|)^n}{n!} = \sum_{n=0}^{\infty}\frac{|x|^n}{n!}\sum_{n=0}^{\infty}\frac{|y|^n}{n!} < \infty.$$

Thus the un-ordered sum is *defined* and *finite.* By using the same calculations now and the same sets A_n that we used when x and y were positive, we get the same result. Finally, if one of x or y is zero, there is really nothing to prove: one of the factor series has value one, and the sum series agrees with the other factor.

§3: A generalization of Theorem (1.9)

(3.1) **Theorem:** *Let* $\sum_{\alpha\in A} a_\alpha$ *be an un-ordered sum of non-negative real numbers.*
If the index set A *is the disjoint union of sets* A_σ, *that is, if* $A = \bigcup_{\sigma\in\Sigma} A_\sigma$, *where* $A_\sigma \cap A_\tau = \emptyset$ *if* $\sigma \neq \tau$, *then* $\sum_{\alpha\in A} a_\alpha = \sum_{\sigma\in\Sigma}\left(\sum_{\alpha\in A_\sigma} a_\alpha\right)$.

The proof is essentially the same as that of Theorem (1.9), using un-ordered sums in place of series!

Example: Calculate $\sum_{n=1}^{\infty}\frac{n(n+1)}{2^n}$. To start we recall that $n(n+1) = 2\sum_{k=1}^{n} k = 2\sum_{k=1}^{\infty}[k \leq n]k$, where $[k \leq n] = 1$ if $k \leq n$ is TRUE, and $[k \leq n] = 0$ if $k \leq n$ is FALSE. We can use this trick (Iverson-Knuth notation) again to write $k = \sum_{j=1}^{\infty}[j \leq k]$. we can then insert all this into our sum to get

$$\sum_{n=1}^{\infty}\frac{n(n+1)}{2^n} = 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{[k \leq n][j \leq k]}{2^n} = 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{[j \leq k]}{2^n}$$
$$= 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{2[j \leq k]}{2^k} = 4\sum_{j=1}^{\infty}\frac{2}{2^j} = 8.$$

What we did here was write the series as a *triple* series, change the order of summation, use geometric series.

Reference

Knuth, Donald, *Two Notes on Notation,* American Math. Monthly 99 (1992) 403 - 426.