Introduction We will prove that if $I \subseteq \bigcup_{k=1}^{K} I_k^*$ then $v(I) \le \sum_{k=1}^{K} v(I_k^*)$. We may assume $v(I) > 0$ (otherwise the desired inequality is trivially true). The intervals I_k^* are not assumed to be non-overlapping (i.e., it may happen that $I_k^* \cap I_{k'}^*$ has non-empty interior). We will use the inequality

$$(*) \qquad \text{if } J \subseteq K, \text{ then } v(J) \le v(K).$$

Proof is immediate.

We will also define $J_k := I_k^* \cap I$, so that $I = \bigcup_{k=1}^{K} J_k$. By $(*)$, all we have to do is prove that $v(I) \le \sum_{k=1}^{K} v(J_k)$.

(Our idea is to express I as the union of finitely many non-overlapping intervals H_j, in such a way that each J_k is the union of certain H_j's. We will see that the H_j "partition" I, so that $\sum_j v(H_j) = v(I)$. We will also see that the H_j "partition" each J_k, so that $\sum_{H_j \subseteq J_k} v(H_j) = v(J_k)$. Since it can happen that some H_j is contained in more than one J_k, the sum of the volumes of the J_k may "count" more than once the volumes of some of the H_j, thus giving a sum larger than the volume of I.)

Some notation What follows is a lot of notation to describe what we have constructed. Let us write

$$I = [a_1, b_1] \times \cdots \times [a_n, b_n] \text{ and, for each } k, \quad J_k = [c_{k1}, d_{k1}] \times \cdots \times [c_{kn}, d_{kn}] \text{ (products of \textit{factor intervals}.)}$$

By construction, for each k and i, $[c_{ki}, d_{ki}] \subseteq [a_i, b_i]$. Moreover, $[a_i, b_i] = \bigcup_k [c_{ki}, d_{ki}]$, for $1 \le i \le n$. Hence the numbers c_{ki} and d_{ki} (for fixed i and for $1 \le k \le K$) can be rearranged to form a partition π_i of $[a_i, b_i]$. We let P_{ij} denote the intervals of π_i, where $1 \le j \le n_i$, with n_i the number of intervals in π_i. We let λ_{ij} denote the length of P_{ij}. Then $b_i - a_i = \sum_{j=1}^{n_i} \lambda_{ij}$. We then have, for each k and i,

$$[c_{ki}, d_{ki}] = \bigcup_{j=j_{ki}}^{j_{ki}+\nu_{ki}-1} P_{ij}, \quad \text{and} \quad d_{ki} - c_{ki} = \sum_{j=j_{ki}}^{j_{ki}+\nu_{ki}-1} \lambda_{ij}$$

(this expresses the fact that each $[c_{ki}, d_{ki}]$ is the union of consecutive intervals P_{ij} from π_i; ν_{ki} is the number of intervals required). Therefore,

$$J_k = \left(\bigcup_{j=j_{k1}}^{j_{k1}+\nu_{k1}-1} P_{1j} \right) \times \cdots \times \left(\bigcup_{j=j_{kn}}^{j_{kn}+\nu_{kn}-1} P_{nj} \right)$$

By the formula (you should check this for your self) $(A \cup B) \times (C \cup D) = (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$, extended to the case of more sets in each union and more factors, with $j = (j_1, \ldots, j_n)$ we have

$$J_k = \bigcup_j P_{1j_1} \times \cdots \times P_{nj_n} \quad \text{(each Cartesian product has } n \text{ factors)}$$

where j runs through all possible values $j = (j_1, \ldots, j_n)$, with $j_{ki} \le j_i \le j_{ki} + \nu_{ki} - 1$, $1 \le i \le n$.

The Cartesian products $P_{1j_1} \times \cdots \times P_{nj_n}$ are our basic building blocks for what I call a *Cartesian partition* of I. We continue with notation for the moment, and define

$$H_j := P_{1j_1} \times \cdots \times P_{nj_n},$$

where j denotes $j = (j_1, \ldots, j_n)$ and where $1 \le j_i \le n_i$ and $1 \le i \le n$. There are $n_1 \cdots n_n$ "values" of j. In particular, $J_k = \cup_j H_j$, where now j runs through all possible values $j = (j_1, \ldots, j_n)$, with $j_{ki} \le j_i \le j_{ki} + \nu_{ki} - 1$, $1 \le i \le n$. There are $\nu_{k1} \cdots \nu_{kn}$ "values" of j involved in J_k.

Partitions and Cartesian partitions A *partition* of an interval $I \subseteq \mathbb{R}^n$ is a finite collection of intervals that are non-overlapping, whose union is I. In more than one dimension, partitions are not nicely ordered in general, so are usually not useful. But one kind of partition *is* useful. A *Cartesian partition* is the collection of intervals induced by *partitions of the factor intervals* of I. For us these are the intervals H_j defined already. To verify that they are non-overlapping, we need another Cartesian product formula, and its extension to n factors:

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

Again, you should check this, and the extension, yourself. By the extended formula, if $j \neq j'$ we have

$$H_j \cap H_{j'} = (P_{1j_1} \cap P_{1j'_1}) \times \cdots \times (P_{nj_n} \cap P_{nj'_n}) \quad \text{(so that } H_j \cap H_{j'} \text{ is an interval).}$$

Since $j_i \neq j'_i$ for at least one i, at least one of the intersections $P_{ij_i} \cap P_{ij'_i}$ is empty or is a singleton. Therefore $H_j \cap H_{j'}$ is empty or is contained in a hyperplane, hence has no interior. Moreover, by definition, $v(H_j \cap H_{j'}) = 0$ since the length of at least one of its factor intervals is zero.

To show that the union of the H_j is I we let $x = (x_1, \ldots, x_n) \in I$. Then for each i, there is some j_i, $1 \le j_i \le n_i$, such that $x_i \in P_{ij_i}$. Hence $x \in P_{1j_1} \times \cdots \times P_{nj_n} = H_j$.

We notice that each J_k has a Cartesian partition, given by $J_k = \cup_j H_j$, where j runs through all possible values $j = (j_1, \ldots, j_n)$, with $j_{ki} \le j_i \le j_{ki} + \nu_{ki} - 1$, $1 \le i \le n$. Let's call the set of these "values" of j V_k.

Why Cartesian partitions are useful: it is possible to "order" the intervals in a Cartesian partition, using the subscripts j that we have defined. The order can be arbitrary, but is usually lexicographic. As an example, we prove that $v(I) = \sum_j v(H_j)$.

$$(**) \qquad v(I) := \prod_{i=1}^{n} (b_i - a_i) = \prod_{i=1}^{n} \sum_{j_i=1}^{n_i} \lambda_{ij_i} = \sum_j \prod_{i=1}^{n} \lambda_{ij_i},$$

where j runs through all the values $j = (j_1, \ldots, j_n)$, with $1 \le j_i \le n_i$, $1 \le i \le n$.

The last equality is the extension to more factors with more terms of the identity $(a+b)(c+d) = ac + ad + bc + bd$. To expand the product we select one term from each factor, multiply all the selected terms together, then add that product to the sum on the right – in all possible ways.

Since $v(H_j) = \prod_{i=1}^{n} \lambda_{ij_i}$, we can rewrite $(**)$ as

$$v(I) = \sum_j v(H_j).$$

Proof of our inequality We need to make use of an observation about $J_k \cap J_{k'}$ in case the intersection has non-empty interior. Suppose that $x = (x_1, \ldots, x_n)$ belongs to the *interior* of $J_k \cap J_{k'}$. Then for each i, there is some j_i, $1 \le j_i \le n_i$, such that $x_i \in P_{ij_i}$. Moreover, each x_i is in the interior of P_{ij_i}, for otherwise x would be a boundary point of J_k or of $J_{k'}$. Therefore $x \in H_j$ for a uniquely determined "value" of j. But then $H_j \subseteq J_k \cap J_{k'}$. We can now define $c(j)$ to be the number of the sets J_k such that $H_j \subseteq J_k$. Then by $(**)$ applied to each J_k,

$$(\dagger) \qquad \sum_k v(J_k) = \sum_k \sum_{j \in V_k} v(H_j) = \sum_j c(j) v(H_j) \ge \sum_j v(H_j) = v(I),$$

since each $c(j) \ge 1$. The two sums with "$\sum_j$" are taken over *all possible* "values" of j. This completes the proof. *The second equality in* $(\dagger)$ *is important! It illustrates a difference between the "Riemann" and "Lebesgue" viewpoints of integration! See the note on pp 80-81 of the text.*