Math 5616H, Spring 2004: *The Weierstrass Approximation Theorem*

We defined $Q_n(x) := c_n(1-x^2)^n$ with c_n chosen so that $\int_{-1}^{1} Q_n(x)\,dx = 1$, and showed that $c_n < n+1$. Given a continuous function $f \in C[0,\,1]$ we replaced it by $g(x) := f(x) - f(0) - x(f(1) - f(0))$, extended to the whole line by setting $g(x) = 0$ if $x \notin [0,\,1]$. This makes g continuous on $\mathbb{R}$, because $g(0) = 0 = g(1)$.

We next examine $Q_n * g(x) := \int_0^1 Q_n(x-t)g(t)\,dt$. We want to show that each $Q_n * g$ is a polynomial, and that $Q_n * g(x) \to g(x)$ *uniformly* for $x \in [0,\,1]$.

Each $Q_n * g$ is a polynomial

$$\begin{aligned}
Q_n * g(x) &= \int_0^1 Q_n(x-t)g(t)\,dt = c_n \int_0^1 (1-(x-t)^2)^n g(t)\,dt = (-1)^n c_n \int_0^1 ((x-t)^2-1)^n g(t)\,dt \\
&= c_n \int_0^1 \sum_{r=0}^{n} \binom{n}{r} (x-t)^{2r} (-1)^r g(t)\,dt = c_n \int_0^1 \sum_{r=0}^{n} \binom{n}{r} (-1)^r \sum_{s=0}^{2r} \binom{2r}{s} x^{2r-s} (-t)^s g(t)\,dt \\
&= c_n \sum_{r=0}^{n} \sum_{s=0}^{2r} \binom{n}{r}\binom{2r}{s} (-1)^{r+s} \left(\int_0^1 t^s g(t)\,dt \right) x^{2r-s}.
\end{aligned}$$

The expression on the last line above is the sum of constants times powers of x, so we can write $P_n(x) := Q_n * g(x)$, a polynomial in x of degree at most $2n$.

Next, to show $Q_n * g(x) \to g(x)$ uniformly for $x \in [0,\,1]$.

An important reason for our choice of c_n was this: $g(x) = g(x)\int_{-1}^{1} Q_n(t)\,dt$. Let $\epsilon > 0$ be given. We want to show that there exists N such that $n \ge N \Rightarrow |P_n(x) - g(x)| < \epsilon$ for all $x \in [0,\,1]$.

We notice now that $P_n(x) = \int_0^1 Q_n(x-t)g(t)\,dt = \int_a^b Q_n(x-t)g(t)\,dt$ for every pair of numbers $a \le 0$ and $b \ge 1$. If we choose $a = x-1 \le 0$, and choose $b = 1+x \ge 1$, we can write $P_n(x) - g(x) = \int_{x-1}^{x+1} Q_n(x-t)g(t)\,dt - g(x)\int_{-1}^{1} Q_n(t)\,dt$. In the first integral, we make a *literal* substitution, replacing t by $x-t$:

$$\int_{t=x-1}^{t=x+1} Q_n(x-t)g(t)\,dt = \int_{x-t=x-1}^{x-t=x+1} Q_n(x-(x-t))g(x-t)\,d(x-t) = \int_{t=1}^{t=-1} Q_n(t)g(x-t)\,(-1)dt.$$

Changing the order of the endpoints on the integral gets rid of the -1 on dt, so we finally have

$$P_n(x) - g(x) = \int_{-1}^{1} Q_n(t)g(x-t)\,dt - g(x)\int_{-1}^{1} Q_n(t)\,dt = \int_{-1}^{1} Q_n(t)(g(x-t) - g(x))\,dt.$$

Therefore, if $0 < \delta < 1$,

$$|P_n(x) - g(x)| \le \int_{-1}^{1} Q_n(t)|g(x-t) - g(x)|\,dt = \int_{-\delta}^{\delta} Q_n(t)|g(x-t) - g(x)|\,dt + \int_{\delta<|t|\le 1} Q_n(t)|g(x-t) - g(x)|\,dt.$$

Since g is uniformly continuous there exists $\delta > 0$ such that $|x-y| < \delta \Rightarrow |g(y) - g(x)| < \epsilon/2$.
Since $Q_n(t) \le c_n(1-\delta^2)^n \le (n+1)(1-\delta^2)^n$ if $\delta \le |t| \le 1$, we see that

$$\begin{aligned}
|P_n(x) - g(x)| &\le \int_{-\delta}^{\delta} Q_n(t)(\epsilon/2)\,dt + \int_{\delta<|t|\le 1} (n+1)(1-\delta^2)^n 2\|g\|_\infty\,dt \\
&\le \int_{-1}^{1} Q_n(t)(\epsilon/2)\,dt + 2(n+1)(1-\delta^2)^n 2\|g\|_\infty < \epsilon
\end{aligned}$$

if $n \ge N$, where N is so large that $4(N+1)(1-\delta^2)^N < \dfrac{\epsilon}{2\|g\|_\infty + 1}$ and such that $(n+1)(1-\delta^2)^n$ decreases for $n \ge N$. Then, for $n \ge N$, $|P_n(x) - g(x)| < \epsilon$ for all $x \in [0,\,1]$.