Math 8602, Spring 2004 A Schur Thing

Further Problem 3: Due Apr 16

Prove another result of Schur: *If* K *is measurable on* $X \times Y$, $\int_X |K(s,\,t)|\,ds \le A$, $\int_Y |K(s,\,t)|\,dt \le B$ *and* $1 < p < \infty$ *then when* $f \in L^p(Y)$ *and* $Kf(s) := \int_Y K(s,\,t)f(t)\,dt$ *we have* $\|Kf\|_{p,ds} \le A^{1/p}B^{1/q}\|f\|_{p,dt}$. Include consideration of the existence of the integrals involved. Deal with the cases $p = 1$ and $p = \infty$ as well.

It was to be taken for granted that X and Y were each measurable subsets (with positive measure) of some Euclidean spaces.

We use the converse of Hölder's Inequality, for more than one reason! We suppose that $f \in L^p(Y)$ and $g \in L^q(X)$ with $\|g\|_q \le 1$. Using Hölder and Tonelli, at first supposing $1 < p < \infty$,

$$\begin{aligned}\int_{X\times Y} |K(s,\,t)||f(t)||g(s)|\,dt\,ds &= \int_{X\times Y} |K(s,\,t)|^{\frac{1}{p}}|f(t)||K(s,\,t)|^{\frac{1}{q}}|g(s)|\,dt\,ds\\ &\le \left\{\int_{X\times Y}|K(s,\,t)||f(t)|^p\,dt\,ds\right\}^{\frac{1}{p}}\left\{\int_{X\times Y}|K(s,\,t)||g(s)|^q\,dt\,ds\right\}^{\frac{1}{q}} \quad \text{(Hölder)}\\ &= \left\{\int_X\left[\int_Y |K(s,\,t)||f(t)|^p\,ds\right]dt\right\}^{\frac{1}{p}}\left\{\int_Y\left[\int_X |K(s,\,t)||g(s)|^q\,dt\right]ds\right\}^{\frac{1}{q}} \quad (\text{Tonelli}^2)\\ &\le A^{\frac{1}{p}}\left\{\int_X |f(t)|^p\,dt\right\}^{\frac{1}{p}}B^{\frac{1}{q}}\left\{\int_Y |g(s)|^q\,ds\right\}^{\frac{1}{q}} \le A^{\frac{1}{p}}B^{\frac{1}{q}}\|f\|_p < \infty.\end{aligned}$$

Again, by Tonelli's Theorem,

$$(*) \qquad \int_{X\times Y}|K(s,\,t)||f(t)||g(s)|\,dt\,ds = \int_X\int_Y |K(s,\,t)||f(t)||g(s)|\,dt\,ds \le A^{\frac{1}{p}}B^{\frac{1}{q}}\|f\|_p < \infty.$$

Thus the function $K(s,\,t)f(t)g(s) \in L(X \times Y)$. Hence by Fubini's Theorem, $\int_Y K(s,\,t)f(t)g(s)\,dt$ exists for a.e. $s \in X$, is measurable there, and in $L(X)$. Since g was arbitrary in the closed unit ball of $L^q(X)$, $\int_Y |K(s,\,t)f(t)|\,dt$ is measurable on X and is finite a.e. in X. Hence $Kf(s)$ exists a.e. in X, is measurable and, by the converse of Hölder's Inequality and $(*)$, is in $L^p(X)$ and satisfies the inequality $\|Kf\|_{p,ds} \le A^{1/p}B^{1/q}\|f\|_{p,dt}$.

The subtle point here is that although $\int_Y |K(s,\,t)f(t)|\,dt$ exists simply because the integrand is measurable, it might be infinite a.e., and in that case would not allow us to say that $\int_Y K(s,\,t)f(t)\,dt$ exists almost everywhere! Thus the finiteness of $\int_Y |K(s,\,t)f(t)|\,dt$ a.e. needs to be established *first*. *Then* we can say that $Kf(s)$ is well-defined for a.e. s.