

(1) **Definition:** The space ℓ^2 consists of all sequences $\{c_n\}_{n \in \mathbb{Z}}$ of complex numbers about which it is true that $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$, together with the inner product $\langle c, d \rangle := \sum_{n=-\infty}^{\infty} c_n \overline{d_n}$. The inner product series converges absolutely because $2|c_n \overline{d_n}| \leq |c_n|^2 + |d_n|^2$ (by Cauchy's Inequality). Sometimes we will write $c(n)$ instead of c_n . The c_n are called the *terms*, or *components*, or *coordinates* of c .

(2) **Exercise:** Show that ℓ^2 is an inner product space. Take it for granted that ℓ^2 is a complex vector space.

We are especially interested in the elements e_n of ℓ^2 defined by $e_n(k) = \delta_{nk}$, for $k \in \mathbb{Z}$. Each e_n , $n \in \mathbb{Z}$, is the sequence indexed by k whose terms are all zero except when $k = n$, and that term is one. They are called the (standard) *unit vectors* in ℓ^2 . We define $\mathcal{B} := \{e_n : n \in \mathbb{Z}\}$, the set of standard unit vectors. It is important to realize that if $c \in \ell^2$ and $d \in \ell^2$ then $c = d$ if and only if $c_n = d_n$ for all $n \in \mathbb{Z}$. Moreover, $c_n = \langle c, e_n \rangle$ for all $n \in \mathbb{Z}$. A sequence $c \in \ell^2$ is a *finite sequence* if all but a finite number of the c_n are zero.

(3) **Exercise:** Show that $\text{span}(\mathcal{B})$ is the set of all finite sequences in ℓ^2 .

(4) **Theorem:** The inner product space ℓ^2 is a Hilbert space and the set of all the e_n is an orthonormal basis.

The proof of this Theorem has some obvious steps and some that are far from obvious! Let us assume for the moment that we have proved that ℓ^2 is a Hilbert space. To then prove that $\mathcal{B} = \{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis will be very easy from the abstract point of view, and fairly easy to prove directly.

The abstract-based proof that \mathcal{B} is an orthonormal basis: Suppose that \mathcal{O} is an orthonormal subset of ℓ^2 that contains \mathcal{B} and at least one other element, ω , that has norm one and that is perpendicular to every element of \mathcal{B} . Then $\omega_n = \langle \omega, e_n \rangle = 0$ for all $n \in \mathbb{Z}$, so $\omega = 0$ (why!?). Thus no such \mathcal{O} can exist. That is, \mathcal{B} is a maximal orthonormal set and hence (by definition) an orthonormal basis.

The direct proof (this has the advantage that it does not use the Axiom of Choice): by Exercise (16.1) in [1], we only have to show that $\text{span}(\mathcal{B})$ is dense in ℓ^2 . Given $N \in \mathbb{Z}^+$ we define $S_N(c) := \sum_{|k| \leq N} c_k e_k$. The components of $S_N(c)$ are c_n when $|n| \leq N$ and they are zero when $|n| > N$. Thus (since the "tails" of a convergent series tend to zero),

$$\|c - S_N(c)\|^2 = \sum_{|k| > N} |c_k|^2 \rightarrow 0 \text{ as } N \rightarrow \infty. \text{ In other words, } c = \sum_{n=-\infty}^{\infty} c_n e_n, \text{ meaning that } \lim_{N \rightarrow \infty} S_N(c) = c.$$

Remark In ℓ^2 the formula $c = \sum_{v \in \mathcal{B}} \langle c, v \rangle v$ (see [1], between (13.1) and (13.2)) has the more "normal" form

$$c = \sum_{n=-\infty}^{\infty} c_n e_n. \text{ That's why the proof was short.}$$

Proof that ℓ^2 is a Hilbert space: Suppose that $\{c_m\}$ is a Cauchy sequence in ℓ^2 . We have to show that there exists $d \in \ell^2$ such that $c_m \rightarrow d$ in ℓ^2 .

First we will find a *candidate* for d . We need to use this fact: a Cauchy sequence of complex numbers has a (unique) limit in \mathbb{C} . We will denote the terms of $c_m \in \ell^2$ by c_{mn} or by $c_{m,n}$ if we have to. The c_{mn} are complex numbers. Then for each n , $|c_{m,n} - c_{k,n}| \leq \|c_m - c_k\| \rightarrow 0$ as $\min\{k, m\} \rightarrow \infty$, so each sequence $\{c_{m,n}\}_{m \in \mathbb{Z}^+}$ is a Cauchy sequence of complex numbers. Thus $\{c_{m,n}\}_{m \in \mathbb{Z}^+}$ has a limit d_n in \mathbb{C} . That is, $d_n = \lim_{m \rightarrow \infty} c_{m,n}$ exists for each $n \in \mathbb{Z}$.

Next we show that $d = \{d_n\}_{n \in \mathbb{Z}}$ is in ℓ^2 . All we really have to do is show that there is a number A such that for every positive integer N , $\sum_{|n| \leq N} |d_n|^2 \leq A$. Then the symmetric partial sums $\sum_{|n| \leq N} |d_n|^2$ would be bounded above,

thus would have a limit $\sum_{n=-\infty}^{\infty} |d_n|^2 \leq A < \infty$. It would then be true that $d \in \ell^2$.

It is not obvious how to show this boundedness. We use an important lemma that we will use many times, without always mentioning it!

(5) **Lemma:** *In a normed space, every Cauchy sequence is bounded (in norm).*

We will use the lemma, and prove it later in this Note. The Lemma assures us that there exists A such that $\|c_m\| \leq A$ for all $m \in \mathbb{Z}$. Thus for each positive integer N and every $m \in \mathbb{Z}^+$,

$$\sum_{|n| \leq N} |c_{m,n}|^2 \leq \sum_{n=-\infty}^{\infty} |c_{m,n}|^2 \leq A^2. \text{ But then } \sum_{|n| \leq N} |d_n|^2 = \lim_{m \rightarrow \infty} \sum_{|n| \leq N} |c_{m,n}|^2 \leq A^2, \text{ as desired.}$$

Once we had the Lemma and the idea to use it, the proof was easy.

Proof of completeness Next we want to show that $\|c_m - d\| \rightarrow 0$ as $m \rightarrow \infty$. It is not at all obvious what to do here! Even when we find our idea to use, it is not at all clear how to make it work. It would be easy to make the idea work if we were in ℓ^1 and not in ℓ^2 . Thus the complication is *technical*. There is one easy case: $c_m = c_K$ for some K . Then we we'd have $d = c_K$ and be done. So we assume that's not true.

Here is the *idea*: For each positive integer k we choose the “first” positive integer m_k such that $r \geq m_k$ and $s \geq m_k$ together imply that $\|c_r - c_s\| < 2^{-k-1}$. This can be done in such a way that $m_{k+1} > m_k$. The numbers m_k exist because our sequence $\{c_m\}$ is Cauchy in ℓ^2 . As we have seen this means that $|c_{r,n} - c_{s,n}| < 2^{-k-1}$ for every $n \in \mathbb{Z}$, if $r \geq m_k$ and $s \geq m_k$. Now the keys: we work with the members c_{m_k} of our Cauchy sequence. Then $\|c_{m_{k+1}} - c_{m_k}\| < 2^{-k-1}$ and so for all $n \in \mathbb{Z}$ $|c_{m_{k+1},n} - c_{m_k,n}| < 2^{-k-1}$. Thus for every positive integer K

$$(6) \quad d_n = c_{m_K,n} + \sum_{k=K}^{\infty} (c_{m_{k+1},n} - c_{m_k,n}), \text{ for this is a “telescoping” series.}$$

Hence

$$(7) \quad |d_n - c_{m_K,n}| \leq \sum_{k=K}^{\infty} |c_{m_{k+1},n} - c_{m_k,n}| \quad \left(< \sum_{k=K}^{\infty} 2^{-k-1} = 2^{-K} \right).$$

If we were in ℓ^1 , we'd be nearly done. We could add the first inequality over all $n \in \mathbb{Z}$ and get

$$(8) \quad \|d - c_{m_K}\|_1 \leq \sum_{k=K}^{\infty} \|c_{m_{k+1}} - c_{m_k}\|_1 < \sum_{k=K}^{\infty} 2^{-k-1} = 2^{-K}.$$

But we cannot square the left-hand side of (7) and put squares inside the sum on the right! We are going to get an inequality like (8), but not so easily! It will use two tools. One of them is a Lemma that gives a new way, a very “practical” way, to look at how we can find a norm (it's theoretical but useful nonetheless). The other, that depends on the Lemma, is an important and useful Theorem called *Minkowski's Series Inequality* that has the intuitive sense that “the norm of a sum is at most the sum of the norms” even if the “sum” is an infinite series. There is a (more common) integral form of the Theorem (see [2, (5)]). To use these tools in the proof of completeness we only use the Theorem. The Lemma is just used to prove the Theorem.

Once again, we will state the Lemma and the Theorem and use them. Proofs will be done later in this Note.

(9) **Lemma:** *For every sequence $\{x_n\}_{n \in \mathbb{Z}}$ of complex numbers, $\sum_{n=-\infty}^{\infty} |x_n|^2$, whether finite or infinite, is given by the expression*

$$(9.1) \quad \sum_{n=-\infty}^{\infty} |x_n|^2 = \sup \left\{ |\langle x, d \rangle|^2 : d \text{ is a finite sequence and } \|d\| \leq 1 \right\}.$$

This requires a little explanation! If a set $S \subseteq \mathbb{R}$ is nonempty and *not* bounded above we say $\sup S = \infty$. This will be covered in the proof.

(10) **Theorem (Minkowski's Series Inequality):** If $\{A_{kn}\}$ is a $\mathbb{Z} \times \mathbb{Z}$ matrix of complex numbers and $\sum_{k=-\infty}^{\infty} |A_{kn}| < \infty$ for all $n \in \mathbb{Z}$, then

$$\left\{ \sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} A_{kn} \right|^2 \right\}^{1/2} \leq \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} |A_{kn}|^2 \right\}^{1/2}.$$

Note that the sums are taken in different order on the two sides of the inequality! The left side is the “norm of a sum,” and the right side is the “sum of (a series of) norms.”

To continue the proof of completeness we recall (6):

$$(11) \quad d_n = c_{m_K, n} + \sum_{k=K}^{\infty} (c_{m_{k+1}n} - c_{m_k n}), \quad \text{or} \quad d_n - c_{m_K, n} = \sum_{k=K}^{\infty} (c_{m_{k+1}n} - c_{m_k n}).$$

To use Minkowski's theorem we define $A_{kn} := (c_{m_{k+1}n} - c_{m_k n})$ if $k \geq K$, and $A_{kn} := 0$ if $k < K$. Then by (7),

$$\sum_{k=-\infty}^{\infty} |A_{kn}| < \infty \quad \text{for all } n \in \mathbb{Z}. \quad \text{Now we have to translate the various sums in (10). From the second part of (11),}$$

$$\sum_{k=-\infty}^{\infty} A_{kn} = \sum_{k=K}^{\infty} A_{kn} = d_n - c_{m_K, n} \quad \text{and the definition of } A_{kn} \text{ gives } \sum_{n=-\infty}^{\infty} |A_{kn}|^2 = \|c_{m_{k+1}} - c_{m_k}\|^2 < 2^{-2k-2}.$$

Thus the inequality in (10) becomes

$$\|d - c_{m_K}\| = \left\{ \sum_{n=-\infty}^{\infty} |d_n - c_{m_K, n}|^2 \right\}^{1/2} \leq \sum_{k=K}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} |A_{kn}|^2 \right\}^{1/2} = \sum_{k=K}^{\infty} \|c_{m_{k+1}} - c_{m_k}\| < 2^{-K}.$$

We are almost done. Let $\epsilon > 0$ be given. We choose K so large that $2^{-K} < \epsilon/2$. Then, if $m \geq m_K$ we have

$$\|d - c_m\| \leq \|d - c_{m_K}\| + \|c_m - c_{m_K}\| < 2^{-K} + 2^{-K} < \epsilon.$$

This shows that $c_m \rightarrow d$ in ℓ^2 . The proof is done now, except for the proofs of two Lemmas and Minkowski's Series Inequality.

Proofs of two lemmas and a theorem

(“5”) **Lemma:** In a normed space, every Cauchy sequence is bounded (in norm).

Proof: If $\{x_n\}$ is a Cauchy sequence in a normed space then (by definition) there exists a positive integer M such that $n \geq M \Rightarrow \|x_n - x_M\| < 1$. Therefore if $n \geq M$, $\|x_n\| \leq \|x_n - x_M\| + \|x_M\| < 1 + \|x_M\|$. If $n < M$ then $\|x_n\| \leq \max\{\|x_1\|, \|x_2\|, \dots, \|x_{M-1}\|\}$. Hence for arbitrary n ,

$$\|x_n\| \leq \max\{\|x_1\|, \|x_2\|, \dots, \|x_{M-1}\|, \|x_M\| + 1\} =: A < \infty.$$

(“9”) **Lemma:** For every sequence $x = \{x_n\}_{n \in \mathbb{Z}}$ of complex numbers, $\sum_{n=-\infty}^{\infty} |x_n|^2$, whether finite or infinite, is given by the expression

$$(9.1) \quad \sum_{n=-\infty}^{\infty} |x_n|^2 = \sup \left\{ |\langle x, d \rangle|^2 : d \text{ is a finite sequence and } \|d\| \leq 1 \right\}.$$

Proof: If d is a finite sequence then for some N (that depends on d), $d_n = 0$ if $|n| > N$. Thus no matter what the sequence x is, $\sum_{|n| \leq N} x_n \bar{d}_n$ makes sense. Thus although $\langle x, d \rangle$ may not make sense (in case x is not in ℓ^2)

we will cheerfully abuse notation and write $\langle x, d \rangle$ for $\sum_{|n| \leq N} x_n \bar{d}_n$.

Then $S_x := \left\{ |\langle x, d \rangle|^2 : d \text{ is a finite sequence and } \|d\| \leq 1 \right\}$ is a set of real numbers that is not empty (it contains 0, for example). If S_x is not bounded then for every R , no matter how large, there is in S_x a number $|\langle x, d \rangle|^2 > R^2$. That means there is a finite sequence d , that depends on R , such that $|\langle x, d \rangle|^2 > R^2$. There is some N that depends on this d such that $\langle x, d \rangle = \sum_{|n| \leq N} x_n \bar{d}_n$. By the Schwarz Inequality, and because $\|d\| \leq 1$,

$$(12) \quad |\langle x, d \rangle| = \left| \sum_{|n| \leq N} x_n \bar{d}_n \right| \leq \left\{ \sum_{|n| \leq N} |x_n|^2 \right\}^{1/2}, \text{ so that } R^2 < \sum_{|n| \leq N} |x_n|^2 \leq \sum_{n=-\infty}^{\infty} |x_n|^2.$$

It follows that $\sum_{n=-\infty}^{\infty} |x_n|^2 = \infty$ (because if $\sum_{n=-\infty}^{\infty} |x_n|^2$ were finite we would choose $R^2 := \sum_{n=-\infty}^{\infty} |x_n|^2$ and prove the contradiction $R^2 < R^2$). And since S_x has no upper bound, we say $\sup S_x = \infty$. We have shown that the equality (9.1) in the Lemma is true if S_x is not bounded above.

In case x is the zero sequence $S_x = \{0\}$ and the lemma holds in this case. We thus assume that $x \neq 0$.

Now let us assume that S_x is bounded above, say by $B^2 < \infty$ (we assume $B > 0$). We now choose N to be a large positive integer, large enough that $x_n \neq 0$ for at least one n with $|n| \leq N$. We define

$$B_N := \left\{ \sum_{|n| \leq N} |x_n|^2 \right\}^{1/2} \quad (\text{so } B_N > 0) \quad \text{and} \quad d_n := \begin{cases} x_n/B_N, & \text{if } |n| \leq N; \\ 0, & \text{if } |n| > N. \end{cases}$$

Then d is a finite sequence and $\|d\| \leq 1$. Note that there is a d for each sufficiently large N . Therefore

$$\langle x, d \rangle = \sum_{|n| \leq N} |x_n|^2/B_N = |\langle x, d \rangle| \leq B, \text{ or } B_N B \geq \sum_{|n| \leq N} |x_n|^2 = B_N^2, \text{ so } B_N \leq B.$$

But $B_N \leq B$ means $\sum_{|n| \leq N} |x_n|^2 = B_N^2 \leq B^2$. Since this is true for all N sufficiently large, $\sum_{n=-\infty}^{\infty} |x_n|^2 \leq B^2 < \infty$.

This means: if S_x is bounded then $x \in \ell^2$ and $\|x\|^2 \leq B^2$. This inequality is true for every upper bound B^2 of our set S_x . It is therefore true for the least upper bound, $\sup S_x$. We have shown “half of” the equality of the Lemma: $\|x\|^2 \leq \sup S_x$.

To show the “other half” we may suppose that $x \in \ell^2$. Then for every finite sequence d such that $\|d\| \leq 1$ the Schwarz Inequality gives us, as in (12),

$$|\langle x, d \rangle|^2 \leq \sum_{|n| \leq N} |x_n|^2 \leq \|x\|^2, \text{ so that } \|x\|^2 \text{ is an upper bound for } S_x.$$

Since every upper bound is at least as large as the supremum of a set, $\sup S_x \leq \|x\|^2$ and the proof of (9) is done.

(“10”) **Theorem (Minkowski’s Series Inequality):** If $\{A_{kn}\}$ is a $\mathbb{Z} \times \mathbb{Z}$ matrix of complex numbers and $\sum_{k=-\infty}^{\infty} |A_{kn}| < \infty$ for all $n \in \mathbb{Z}$, then

$$\left\{ \sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} A_{kn} \right|^2 \right\}^{1/2} \leq \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} |A_{kn}|^2 \right\}^{1/2}.$$

Proof: We will use (9) and some calculations in the proof. The basic tool is simple: if we have a double series that is a finite sum in one of the variables, we can interchange the order of summation and have the same value either way. We bring in (9) so that we can make sure our double sum has one variable in which the sum is finite.

We begin the work now. First we set $x_n := \sum_{k=-\infty}^{\infty} A_{kn}$ (good technique: give a simple name to a complicated quantity). By hypothesis x_n is a complex number because the series defining x_n converges absolutely. We will switch back and forth between the sums and the x_n 's. By (9)

$$(13) \quad \sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} A_{kn} \right|^2 = \sum_{n=-\infty}^{\infty} |x_n|^2 = \sup \left\{ |\langle x, d \rangle|^2 : d \text{ is a finite sequence and } \|d\| \leq 1 \right\}.$$

It will be convenient to notice that here the set we are taking the supremum of is a set of non-negative numbers. We can thus equally well write

$$\left\{ \sum_{n=-\infty}^{\infty} |x_n|^2 \right\}^{1/2} = \sup \left\{ |\langle x, d \rangle| : d \text{ is a finite sequence and } \|d\| \leq 1 \right\}.$$

We now choose some finite sequence d with $\|d\| \leq 1$. As before we select a positive integer M such that $d_n = 0$ for $|n| > M$. Then, as we said before the work began,

$$(14) \quad \langle x, d \rangle = \sum_{|n| \leq M} x_n \bar{d}_n = \sum_{|n| \leq M} \left(\sum_{k=-\infty}^{\infty} A_{kn} \right) \bar{d}_n = \sum_{k=-\infty}^{\infty} \left(\sum_{|n| \leq M} A_{kn} \bar{d}_n \right).$$

By the Schwarz Inequality and $\|d\| \leq 1$

$$(15) \quad \left| \sum_{|n| \leq M} A_{kn} \bar{d}_n \right| \leq \left\{ \sum_{|n| \leq M} |A_{kn}|^2 \right\}^{1/2} \left\{ \sum_{|n| \leq M} |d_n|^2 \right\}^{1/2} \leq \left\{ \sum_{|n| \leq M} |A_{kn}|^2 \right\}^{1/2} \leq \left\{ \sum_{n=-\infty}^{\infty} |A_{kn}|^2 \right\}^{1/2}.$$

Hence, from (14) and the (series!) triangle inequality and (15) we see that

$$|\langle x, d \rangle| \leq \sum_{k=-\infty}^{\infty} \left| \sum_{|n| \leq M} A_{kn} \bar{d}_n \right| \leq \sum_{k=-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{\infty} |A_{kn}|^2 \right\}^{1/2}.$$

Finally, we use (9) and (13) to complete the proof.

The connection with [3]: Multiresolution Analysis (MRA) to scaling function

In [3, page one] we introduced the Box function $B(x)$. Its translates $B(x - n)$, $n \in \mathbb{Z}$, are functions in $L^2(\mathbb{R})$ and the set of these box functions is an orthonormal set in $L^2(\mathbb{R})$ that can be matched with our set \mathcal{B} that we showed to be an orthonormal basis for ℓ^2 ; we match $B(x - n)$ and e_n . Via this matchup we also can say that every $f \in V_0$ is matched with a $c \in \ell^2$ by just matching coefficients: $f(x) = \sum_{n=-\infty}^{\infty} c_n B(x - n)$ is paired with

$\{c_n\}_{n \in \mathbb{Z}}$, $g(x) = \sum_{n=-\infty}^{\infty} d_n B(x - n)$ is paired with $\{d_n\}_{n \in \mathbb{Z}}$. The useful thing: $\langle f, g \rangle = \langle c, d \rangle$. This allows us to easily justify many statements, including: V_0 is a closed subspace of $L^2(\mathbb{R})$.

References

- [1] *A Brief Introduction to Inner Product Spaces and Hilbert Spaces*, class Web page.
- [2] *The Lebesgue facts*, class Web page.
- [3] *Multiresolution Analysis (MRA) to scaling function*, class Web page.