Introduction We suppose $E \subseteq \mathbb{R}^n$ is measurable and $f : E \to [0, +\infty]$. We do not suppose that f is measurable. We suppose instead that $\overline{R}(f, E) := \{(x, y) \in E \times \mathbb{R} : 0 \le y < f(x)\} \cup \{(x, f(x)) \in E \times \mathbb{R} : f(x) < +\infty\}$ is measurable. We will prove the necessity part of Theorem (5.1), namely that under the assumptions we have made, f is measurable. Here, $|S|$ denotes measure in $\mathbb{R}^{n+1}$ and $|S|_n$ denotes measure in $\mathbb{R}^n$.

First steps Given $\alpha \in \mathbb{R}$ we wish to show that $\{f > \alpha\}$ is measurable. We will actually prove that $\{f \ge \alpha\}$ is measurable. If $\alpha \le 0$, $\{f \ge \alpha\} = E$, so we assume that $\alpha > 0$. Let us set $B_\alpha := \overline{R}(f, E) \cap (E \times [0, \alpha])$, and observe that B_α is measurable. Next we construct the diagonal $(n+1) \times (n+1)$ matrix D_α whose entries are all 1 except for the last, which is $1/\alpha$. Then

$$C_\alpha := D_\alpha B_\alpha = \{(x, y) \in E \times \mathbb{R} : 0 \le y \le \min\{1, f(x)/\alpha\}\ \}$$

is measurable and for all $x \in E$, $\{x\} \times [0, 1] \subseteq C_\alpha$ if and only if $f(x) \ge \alpha$. We will now write I for $[0, 1]$.

Third step: the construction of a Lipschitz transformation $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ designed to partially "shrink" the set C_α. Repeated application of T will allow us to deduce the measurability of $\{f \ge \alpha\} \times I$.

We begin by defining $\varphi : \mathbb{R} \to I$ by $\varphi(u) := (u \vee 0)^2 \wedge 1$ and proving that $|\varphi(u) - \varphi(v)| \le 2|u - v|$ for all real u and v:

If u is	v is	then $\lvert u-v\rvert =$	$\lvert\varphi(u) - \varphi(v)\rvert =$	which is
≤ 0	≤ 0	$\lvert u-v\rvert$	0	$\le \lvert u-v\rvert$
≤ 0	in $(0,1)$	$\lvert u\rvert + v$	v^2	$\le v \le \lvert u-v\rvert$
≤ 0	≥ 1	$\lvert u\rvert + v$	1	$\le v \le \lvert u-v\rvert$
in $(0,1)$	≤ 0	$u + \lvert v\rvert$	u^2	$\le u \le \lvert u-v\rvert$
in $(0,1)$	in $(0,1)$	$\lvert u-v\rvert$	$\lvert u^2 - v^2\rvert$	$= \lvert u-v\rvert(u+v) < 2\lvert u-v\rvert$
in $(0,1)$	≥ 1	$v - u$	$1 - u^2$	$= (1-u)(1+u) < 2(1-u) \le 2\lvert u-v\rvert$
≥ 1	≤ 0	$u + \lvert v\rvert$	1	$\le u \le \lvert u-v\rvert$
≥ 1	in $(0,1)$	$u - v$	$1 - v^2$	$= (1-v)(1+v) < 2(1-v) \le 2\lvert u-v\rvert$
≥ 1	≥ 1	$\lvert u-v\rvert$	0	$\le \lvert u-v\rvert$.

This can be proved more quickly using Calculus, but this proof is more direct.

We put $T(x, y) := (x, \varphi(y))$ and observe that $|T(x, y) - T(x', y')| \le 2|(x, y) - (x', y')|$.

Fourth step We know that therefore $H_1 := T(C_\alpha)$ is measurable, and we define $H_{m+1} := T(H_m)$ for $m \ge 1$. We observe that each H_m is measurable and that for all $x \in E$ and for all m, $\{x\} \times I \subseteq H_m$ if and only if $f(x) \ge \alpha$. If $f(x) < \alpha$, then $(x, y) \notin H_m$ for all sufficiently large m. We then have:

$$H := \bigcap_m H_m \text{ is measurable and } H = (\{f \ge \alpha\} \times I) \cup (E \times \{0\}).$$

Since $E \times \{0\}$ is a set of measure zero, we conclude that $P_\alpha := \{f \ge \alpha\} \times I$ is measurable.

The last step is to show that $\{f \ge \alpha\}$ is measurable. We begin by obtaining a given $\epsilon > 0$ and an open set $\mathcal{G} \subseteq \mathbb{R}^{n+1}$ such that $|\mathcal{G} \setminus P_\alpha| < \epsilon$. For every $x \in \{f \ge \alpha\}$, the compact set $\{x\} \times I \subseteq P_\alpha \subseteq \mathcal{G}$ so there exists $\delta(x) > 0$ such that $B_{\delta(x)}(x) \times I \subseteq \mathcal{G}$. The open set

$$G := \bigcup_{x \in \{f \ge \alpha\}} B_{\delta(x)}(x) \subseteq \mathbb{R}^n \text{ satisfies } \{f \ge \alpha\} \subseteq G \text{ and } P_\alpha \subseteq G \times I \subseteq \mathcal{G}.$$

Thus $\epsilon > |(G \times I) \setminus P_\alpha| = |(G \setminus \{f \ge \alpha\}) \times I|$.

We can now obtain an open set $\mathcal{U} \subseteq \mathbb{R}^{n+1}$ such that $|\mathcal{U} \setminus [(G \setminus \{f \ge \alpha\}) \times I]| < \epsilon$. For every $x \in G \setminus \{f \ge \alpha\}$, the compact set $\{x\} \times I \subseteq \mathcal{U}$ so there exists $\eta(x) > 0$ such that $B_{\eta(x)}(x) \times I \subseteq \mathcal{U}$.

The open set

$$U := \bigcup_{x \in \{f \ge \alpha\}} B_{\eta(x)}(x) \subseteq \mathbb{R}^n \text{ satisfies } G \setminus \{f \ge \alpha\} \subseteq U \text{ and } U \times I \subseteq \mathcal{U}.$$

We can write U as the union of countably many non-overlapping closed dyadic cubes Q_k contained in $\mathbb{R}^n$,

$$U = \bigcup_k Q_k, \text{ so that } \bigcup_k (Q_k \times I) \subseteq \mathcal{U}.$$

But then $|G \setminus \{f \ge \alpha\}|_{e,n} \le |U|_n = \sum_k v_n(Q_k) = \sum_k v_{n+1}(Q_k \times I) < |\mathcal{U}| < \epsilon$, so $\{f \ge \alpha\}$ is measurable.