Here are some ways to look at $\limsup_{n\to\infty} x_n$, when $\{x_n\}$ is a sequence of real numbers. We will apply this when $n \geq 1$ and $x_n = |c_n|^{1/n}$. But to begin we consider arbitrary sequences of real numbers.

(1) **Definition:** *Given a sequence* $\{x_n\}$ *of real numbers we define*

$$\limsup_{n\to\infty} x_n := \lim_{m\to\infty} A_m, \quad \textit{where} \;\; A_m := \sup\{x_n : n \geq m\}.$$

Remarks:

(2) We note that the sequence $\{A_m\}$ decreases (not necessarily strictly!). Thus $\lim_{m\to\infty} A_m = \inf_m A_m$. In the present case, A_m could tend to $-\infty$. For example, if $x_n = -n$ then $A_m = -m$.

(3) It is possible to have $A_m = +\infty$. For example, if $x_n = (-1)^n n$ then $A_m = +\infty$ for every m. In this example, then, $\limsup_{n\to\infty}(-1)^n n = +\infty$. The same is true for *any* sequence that is not bounded above.

(4) But if $\{x_n\}$ is bounded above, A_m is finite for every m, even if $\{x_n\}$ is *not* bounded below: if $x_n = 0$ if n is even, and $x_n = -n$ when n is odd, $A_m = 0$ for every m.

(5) If $\{x_n\}$ converges to x or even if $\{x_n\}$ has an infinite limit, then $\limsup_{n\to\infty} x_n = x$ (and now x can be infinite). To see this we use an approach that includes the infinite cases: if y is a real number and $y < x$ then eventually $x_n > y$. Thus eventually $A_m > y$ because (important!) $A_m \geq x_m$ for every m. Of course, if $x = -\infty$, there is no $y < x$, but then our reasoning is OK vacuously. And if $y > x$ then eventually $x_n < y$ and thus $A_m \leq y$ eventually. Hence $A_m \to x$. Note that by using $x_n \to x$ we do not mean to say that $\{x_n\}$ *converges* to x, for that would connote that x is finite. Our approach is consistent with what we mean when we write $x_n \to +\infty$ or $x_n \to -\infty$.

Other versions of the definition

(6) **Theorem:** *Given a sequence* $\{x_n\}$ *of real numbers we define*

$$SSL(\{x_n\}) := \{y \in [-\infty, +\infty] : \;\; \textit{There exists a subsequence} \;\; x_{n_k} \to y \;\}.$$

SSL is called the subsequential limit set. Then $\limsup_{n\to\infty} x_n = \sup SSL$.

To apply this Theorem we use part of its proof: there exists a subsequence $x_{n_k} \to \sup \mathrm{SSL}$.

(7) **Theorem:** *Given a sequence* $\{x_n\}$ *of real numbers we define*

$$U(\{x_n\}) := \{y \in [-\infty, +\infty] : \;\; \textit{Eventually,} \;\; x_n < y \;\}. \;\; \textit{Then} \;\; \limsup_{n\to\infty} x_n = \inf U(\{x_n\}).$$

To apply this Theorem we suppose that $y > \limsup_{n\to\infty} x_n$. Then we know that eventually eventually $x_n < y$.

Examples to study Find R defined by $\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n}$ for each power series:

$$\sum_{n=1}^{\infty} n^k z^{k^2 n}, \quad (k \text{ a nonzero integer}); \qquad \sum_{n=0}^{\infty} n^2 z^{2^n}; \quad \sum_{n=0}^{\infty} 2^n z^{n^2}; \quad \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n; \quad \sum_{n=0}^{\infty} 2^{n^2} z^{2^n}.$$

Recommended

Modify (1) - (7) to deal with $\liminf_{n\to\infty} x_n$ and show that $x_n \to x$ if and only if $\limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$.