Background

Two formulas are very important in this course. The first (The Binomial Theorem) can be proved by induction, the second (Difference-of-Powers Formula) by direct calculation, using summations.

$$\text{For all real numbers } a \text{ and } b, \text{ and for all } n \in \mathbb{N}, \quad (a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r,$$

$$\text{where } \binom{n}{r} := \frac{n!}{r!(n-r)!} \text{ and where } 0! := 1 \text{ and } n! := n \cdot (n-1)! \text{ for all } n \in \mathbb{N} \text{ such that } n > 1.$$

For later use, let us notice that (by computing them) $\binom{n}{1} = n = \binom{n}{n-1}$.

$$\text{For all real numbers } a \text{ and } b, \text{ and for all } n \in \mathbb{N}, \quad a^n - b^n = (a-b) \sum_{r=0}^{n-1} a^{n-1-r} b^r.$$

Both of these formulas are used in the proof of the existence of n-th roots.

The n-th root theorem

Theorem: Let $n \in \mathbb{Z}^+$. For all real y, if $y \geq 0$ then there exists one and only one non-negative number x such that $x^n = y$.

Proof: First we will prove existence, then uniqueness. In an existence proof we usually expect to use the Completeness Axiom, or some Theorem that uses the Completeness Axiom in its proof.
Notation: We denote our $x \geq 0$ that solves $x^n = y$ by $\sqrt[n]{y}$.

There is a very easy case: $y = 0$. If $y = 0$ we can choose $x = 0$, and we find by induction that $x^n = 0$, so $x = 0$ is a solution of the equation $x^n = 0$. Why is $x = 0$ the *only* solution of $x^n = 0$? If it were true that $x \neq 0$ and $x^n = 0$, then the set $E_x := \{k \in \mathbb{Z}^+ : x^k = 0\}$ would be non-empty. In particular, we could not have $n = 1$, for then $0 = x^1 = x \neq 0$, a contradiction. By the Well-Ordering Property on $\mathbb{N}$, our set E_x would have a least element, m. As we know, $m > 1$, so $m - 1 \in \mathbb{N}$ and thus $x^{m-1} \neq 0$. But then $0 = x^m = x^{m-1}x$, which is the product of two non-zero real numbers. Such a product cannot be zero, so we again have a contradiction. Thus 0 is the unique n-th root of 0. The case $n = 1$ is even easier, so it is left to you to "do in your head."

To continue, we may suppose in what follows that $y > 0$ and $n > 1$.
We will use this set: $S := \{t \in \mathbb{R} : t \geq 0 \text{ and } t^n < y\}$.
The idea is to show that $\sigma = \sup S$ exists, and then prove that $\sigma^n = y$.

Proof that $\sup S$ exists

First, we can show that $S \neq \emptyset$ by guessing a number that is in S. Here is a guess (that depends on the Binomial Theorem): $t := y/(y + \frac{1}{n})$. We have $t^n = \frac{y^n}{\sum_{r=0}^{n} \binom{n}{r} y^{n-r}(1/n)^r}$. If we can show that the denominator is *greater* than y^{n-1} then $t^n < \frac{y^n}{y^{n-1}} = y$. The denominator includes the term obtained by setting $r = 1$, which is $\binom{n}{1} y^{n-1}(1/n)^1 = ny^{n-1}(1/n) = y^{n-1}$. Hence the denominator, which consists of positive terms in addition to y^{n-1}, is greater than y^{n-1}. Therefore

$$t^n = \frac{y^n}{\sum_{r=0}^{n} \binom{n}{r} y^{n-r}(1/n)^r} < \frac{y^n}{y^{n-1}} = y.$$

Thus $t \in S$ so S is not empty.

Next, we need to show that S is bounded above. Let's show that $\frac{y}{n} + 1$ is an upper bound for S. Instead of showing that $t \in S \Rightarrow t < \frac{y}{n} + 1$, let us show the contrapositive, namely $t \geq \frac{y}{n} + 1 \Rightarrow t \notin S$. We see that $t > 0$ if $t \geq \frac{y}{n} + 1$, so to show that such $t \notin S$ we must show that $t^n \geq y$. We use the Binomial Theorem, and the term with $r = n - 1$:

$$t^n \geq \left(\frac{y}{n} + 1\right)^n = \sum_{r=0}^{n} \binom{n}{r} \left(\frac{y}{n}\right)^{n-r} 1^r > \binom{n}{n-1} \left(\frac{y}{n}\right)^{n-(n-1)} = n(y/n) = y.$$

so t cannot belong to S. The contrapositive is true, so the original statement is true. We have proved that S is non-empty and bounded above.

By the Completeness Axiom, S therefore has a least upper bound, $\sup S$, and we put $\sigma := \sup S$.

Proof that $\sigma^n = y$

If we can prove that the statements $\sigma^n > y$ and $\sigma^n < y$ are false, then by Trichotomy, $\sigma^n = y$.

We have made use of the **Power Fact:** if $n \in \mathbb{Z}^+$ and $0 < s < t$, then $s^n < t^n$. Let us make an Observation about S that will be useful. The Observation uses the Power Fact.

Observation: If $0 < s < t$ and $t \in S$, then $s \in S$. This is a "special" property of S. Similarly, if $0 < s < t$ and $s \notin S$, then $t \notin S$. To verify this, we have $0 < s^n < t^n < y$ by the Power Fact and the criterion for membership in S. Thus $s \in S$. For a *positive* s to fail to belong to S, it must be true that $s^n \geq y$, i.e., the membership criterion must be violated by s. Then by the Power Fact, $y \leq s^n < t^n$, so the membership criterion is violated by t as well.

Proof that $\sigma^n > y$ **is false**

Suppose that $\sigma^n > y$. The idea is, by subtracting a positive number h_1 from σ, to construct a number $\sigma - h_1 < \sigma$ that is not in S. Then no member of S is greater than $\sigma - h_1$, by the Observation. This will contradict the property of the supremum that asserts the existence of $s \in S$ such that $\sigma - h_1 < s$, since $\sigma - h_1 < \sup S$.

We start with $0 < h < \sigma$ and use the Difference-of-powers formula and the fact that $(\sigma - h)^r < \sigma^r$ if $r \geq 1$:

$$\sigma^n - (\sigma - h)^n = (\sigma - (\sigma - h)) \sum_{r=0}^{n-1} \sigma^{n-1-r}(\sigma - h)^r = h \sum_{r=0}^{n-1} \sigma^{n-1-r}(\sigma - h)^r < (\sigma - (\sigma - h)) \sum_{r=0}^{n-1} \sigma^{n-1-r}(\sigma)^r = h(n\sigma^{n-1}).$$

We now pick $h_1 := \dfrac{\sigma^n - y}{n\sigma^{n-1}}$. Then $h_1(n\sigma^{n-1}) = \frac{\sigma^n - y}{n\sigma^{n-1}}(n\sigma^{n-1}) = \sigma^n - y$. Hence, if $h = h_1$,

$$\sigma^n - (\sigma - h)^n < h_1(n\sigma^{n-1}) = \sigma^n - y.$$

Therefore $(\sigma - h)^n > y$, so $\sigma - h \notin S$. This gives us the contradiction we wanted.

Proof that $\sigma^n < y$ **is false**

We know that $0 < y/(y + \frac{1}{n}) \leq \sigma$, so $\sigma > 0$, meaning that $\sigma \neq 0$. Therefore, assuming that $\sigma^n < y$ means that $\sigma \in S$. The idea is to show that we can find $h > 0$ so that $\sigma + h \in S$, giving an element of S that is larger than the upper bound σ. This will give us the contradiction that we need.

We set $\epsilon := y - \sigma^n > 0$. Then by the Difference-of-powers formula, for all h such that $0 < h \leq 1$,

$$(\sigma + h)^n - \sigma^n = h \sum_{r=0}^{n-1} (\sigma + h)^{n-1-r}\sigma^r \leq h \sum_{r=0}^{n-1} (\sigma + 1)^{n-1} = h(n(\sigma + 1)^{n-1}).$$

In the first sum, we replaced both of $\sigma + h$ and σ by $\sigma + 1$. Next, let $0 < h < \min\left(1, \dfrac{y - \sigma^n}{n(\sigma + 1)^{n-1}}\right)$. Then

$$(\sigma + h)^n - \sigma^n \leq h(n(\sigma + 1)^{n-1}) < y - \sigma^n, \quad \text{so} \quad (\sigma + h)^n < y.$$

This means $\sigma + h \in S$, and gives the desired contradiction. This completes the proof of existence.

Proof of uniqueness: If $x > 0$ and $\sigma > 0$ and $x^n = \sigma^n = y > 0$, then $0 = (x - \sigma)\sum_{r=0}^{n-1} x^{n-1-r}\sigma^r$. Hence at least one of the factors is zero. Since $x > 0$ and $\sigma > 0$, the second factor is positive (it exceeds x^{n-1}). Thus $x = \sigma$. We already dealt with the case $\sigma = 0$.

This completes the proof of the Theorem.