Ask! Indicate your approach! Show your work!

Quiz 2: June 26.

(1) [15] Write out the definition of "$f(x)$ has a limit at $x = a$," in "logic" format, showing all needed quantifiers in their proper places.

(2) [5] Find $\lim_{x\to 2} \dfrac{x^3 - 4x^2 + 3x + 2}{x - 2}$. Hint: use long division.

When you do the long division you get $x^2 - 2x - 1$, which (say why) has limit -1 at $x = 2$, so the limit is -1.

(3) [20] Prove that your answer to #2 is correct.

This amounts to *proving* that $\lim_{x\to 2} x^2 - 2x = 0$. (say why) Since $x^2 - 2x = x(x - 2)$, we can now use limit theorems (say which ones)

(4) [15] Prove that $x^2 = 3$ has no rational solutions.

Note: In the next two questions, there are no formulas to work with so you have to work with Definitions. But you can also work with Theorems!

(5) [15] Suppose that

$$\lim_{x\to a} g(x) = 0;$$
$$g(x) \neq 0 \text{ if } x \neq a;$$
$$R := \lim_{x\to a} \frac{f(x)}{g(x)} \text{ exists.}$$

Prove that $\lim_{x\to a} f(x) = 0$.

(6) [15] In #5, suppose in addition that $R \neq 0$. Prove that $\lim_{x\to a} \dfrac{g(x)}{f(x)}$ exists. What is the value of the limit?

Quiz 1: June 19.

(1) [10] Prove that $1/(1/a) = a$ for all real $a \neq 0$.

We know (say why) $1/a = a^{-1} \neq 0$, so $1/(1/a) = (a^{-1})^{-1}$. Then (say why) $a^{-1}(a^{-1})^{-1}$ = what? so (say why) $a = 1/(1/a)$.

(2) [15] Prove that $1 \in P$.

Suppose not. Then (say why) $-1 \in P$. Then (say why) $(-1)(-1) = 1 \in P$. This (say why) contradicts (what), so (say why) $1 \in P$.

(3) [20] Prove that if $ab = 0$ then $a = 0$ or $b = 0$.

We can use # 4, below, and thus prove $(\sim B) \Rightarrow (\sim A)$ instead, namely: if $a \neq 0$ and $b \neq 0$ then $ab \neq 0$.

Suppose, instead, that $a \neq 0$ and $b \neq 0$ and $ab = 0$. Then

$$0 = ab = \text{(say why) } (ab)(a^{-1}b^{-1}) = \text{(say why) } (aa^{-1})(bb^{-1}) == \text{(say why) } 1,$$

a contradiction (say why)

(4) [10] Prove that $(A \Rightarrow B) \iff ((\sim B) \Rightarrow (\sim A))$ is a tautology.

Write out the truth table.

Note: $(\sim B) \Rightarrow (\sim A)$ is called the *contrapositive* of $A \Rightarrow B$.

(5) [5] State the contrapositive of #3. Is there any need to prove it, assuming #3 has been proved?

No, as long as # 3 has been proved. However, we proved # 3 by proving its contrapositive! And the contrapositive of the contrapositive of an implication is that implication.

(6) [10] Let $f(x)$ be defined for $x \neq 1$ by $f(x) := \dfrac{x^2-1}{x-1}$. Find L so that, if we let $f(1) := L$, then $f(x)$ is continuous at $x = 1$.

$$\text{If } \ x \neq 1, \quad f(x) = x + 1 = (x-1) + 2, \quad \text{so } \ f(x) - 2 = x - 1.$$

Thus if we choose $L = 2$, we can be given any $\epsilon > 0$ and choose $\delta = \epsilon$ to satisfy the definition of continuity of $f(x)$ at $x = 1$ (say why)

(7) [20] Prove Theorem 2.6.

First Hints: You don't have to use the author's hints. Step 1: Apply the Given info about $g(x)$, setting $\epsilon_1 := |M|/2$, and find $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$, then $|g(x)-M| < |M|/2$. Say why this implies that, if $0 < |x-a| < \delta_1$, then $|g(x)| > |M|/2$. The reason for doing this is that it allows us to "clean up" the denominator, using inequalities to convert quantities that can vary into constants.