Introduction Given the permutation $k_i : \mathbb{Z}^+ \to \mathbb{Z}^+$, $T : [0, 1] \to [0, 1]$ is defined by $T(x) = \sum_{i=1}^{\infty} \frac{\alpha_{k_i}}{2^i}$, where $x = \sum_{i=1}^{\infty} \frac{\alpha_i}{2^i}$, $\alpha_i \in \{0, 1\}$, is the binary representation of x written as $x = .\alpha_1\alpha_2 \cdots \alpha_i \cdots$ that (unless $x = 0 = .\overline{0}$) ends with an infinite string of ones (written $\cdots 0\overline{1}$) rather than an infinite string of zeroes. Only a denumerable subset of $[0, 1]$ has more than one possible representation. When we let ℓ_j denote the prtmutation inverse to k_i we can write $T(x) = \sum_{j=1}^{\infty} \frac{\alpha_j}{2^{\ell_j}}$. With these conventions in place $T(0) = 0$ and $T(1) = 1$.

By construction T is onto. Suppose $T(x) = T(x')$, or $\sum_{i=1}^{\infty} \frac{\alpha_{k_i}}{2^i} = \sum_{i=1}^{\infty} \frac{\alpha'_{k_i}}{2^i}$. If one of x and x' is zero and $T(x) = T(x')$ then the other must be zero as well. Otherwise, if $\{\alpha_{k_i}\} \neq \{\alpha'_{k_i}\}$ there is a first i such that $\alpha_{k_i} \neq \alpha'_{k_i}$. We may suppose $\alpha_{k_i} = 1$ and $\alpha'_{k_i} = 0$. To maintain equality of the sums, we must have $\alpha'_{k_j} = 1$ and $\alpha_{k_j} = 0$ for all $j > i$. This contradicts the requirement that each string have infinitely many ones. Hence T is one–to–one.

First steps We begin by examining the effect of T on a dyadic interval $(\frac{s}{2^K}, \frac{s+1}{2^K}]$, where $0 \leq s < 2^K$. We may assume $K > 0$ since T is onto. For any x in this interval $x = \frac{s}{2^K} + \xi$, where $0 < \xi \leq \frac{1}{2^K}$ and $\xi = .0 \cdots 0\beta_{K+1}\beta_{K+2} \cdots \beta_{K+i} \cdots$, with a string of K leading zeros, not all the β_{K+i} being zero. Our binary representation for $\frac{s}{2^K}$ depends on the binary representation of the integer s as

$$s = \sum_{r=\rho}^{\mu} \epsilon_r 2^r, \quad \epsilon_r \in \{0, 1\}, \quad 0 \leq \rho \leq \mu < K, \quad \text{with} \quad \epsilon_\rho = 1 = \epsilon_\mu.$$

We then have $\frac{s}{2^K} = .0 \cdots 01\epsilon_{\mu-1} \cdots \epsilon_{\rho+1}0\overline{1}$, where there are $K - 1 - \mu$ leading zeros, ϵ_r is in bit position $K - r$, except that ϵ_ρ is replaced by the string $0\overline{1}$, terminating the representation. We observe that the representation for $\frac{s+1}{2^K}$ differs by having a one in bit position $K - \rho$, followed by ρ zeros and then all ones.

When we seek the representation for $x = \frac{s}{2^K} + \xi$ we note that we can use the *terminating* representation for $\frac{s}{2^K}$ and simply write in the bits for ξ, where they belong, since $\xi > 0$. Thus

$$x = \frac{s}{2^K} + \xi = .0 \cdots 01\epsilon_{\mu-1} \cdots \epsilon_{\rho+1}10 \cdots 0\beta_{K+1}\beta_{K+2} \cdots \beta_{K+i} \cdots$$

where the second substring of zeros has ρ of them.

At last, we can write, for K and $s < 2^K$ given (so that ρ and μ are defined),

$$T(x) = \frac{1}{2^{\ell_{K-\mu}}} + \sum_{r=K-\mu+1}^{K-\rho-1} \frac{\epsilon_{K-r}}{2^r} + \frac{1}{2^{K-\rho}} + \sum_{i=1}^{\infty} \frac{\beta_i}{2^{\ell_{i+K-\rho+1}}}.$$

Hence $T(\,(\frac{s}{2^K}, \frac{s+1}{2^K}[\,)$ is a translate of the set T_{sK} of numbers of the form $\sum_{i=1}^{\infty} \frac{\beta_i}{2^{\ell_{i+K-\rho+1}}}$, where the $\beta_i \in \{0, 1\}$.

Third step: We want to show that $|T_{sK}| = 1/2^K$. The numbers in the set T_{sK} have zeros in K different bit positions. Bit position p "partitions" $(0, 1]$ into 2^p disjoint intervals of the form $(\frac{m}{2^p}, \frac{m+1}{2^p}]$, $0 \leq m < 2^p$. If m is even, the numbers in that interval have a zero in bit position p, and all the others have a one in position p (we are ignoring 0). Thus the set with zero in position p is a Borel set and its measure is $1/2$. If we require that another, later, bit position have a zero, then half the intervals due to that position are contained in the intervals that have a zero in position p and half are disjoint from the intervals that have a zero in position p. Thus the set of numbers with zero bits in two different positions is a Borel set with measure $1/4$, so by an induction we can show that T_{sK} is a Borel set and $|T_{sK}| = 1/2^K$, so $|T(\,(\frac{s}{2^K}, \frac{s+1}{2^K}[\,)| = 1/2^K$. Since T is one–to–one (we will use this again and again here without further mention) we have $|T(\,[\frac{s}{2^K}, \frac{s+1}{2^K}[\,)| = 1/2^K$ as well.

The concluding steps A relatively open set can be written as the non-overlapping union of a sequence of dyadic intervals, so T maps open sets (hence also closed sets) to measurable sets and preserves their measures. Given a measurable $E \subseteq [0, 1]$ and $\epsilon > 0$ a closed set $F \subseteq E$ and an open set $G \supseteq (E \setminus F)$ exist such that $|E \setminus F| < \epsilon$ and $|G| < \epsilon$. Then $T(E) \setminus T(F) = T(E \setminus F) \subseteq T(G)$ so $|T(E) \setminus T(F)|_e < \epsilon$. It follows that $T(E)$ is measurable. Moreover, if $|E| < \epsilon$, we can find an open $G \supseteq E$ with $|G| < \epsilon$. Thus $|T(E)| < \epsilon$. Then $|T(E)| = |T(E) \setminus T(F)| + |T(F)| < \epsilon + |F|$ and it follows that $|T(E)| = |E|$. (Some minor steps have been omitted in this note)