Background

Two formulas are very important in this course. The first (The Binomial Theorem) can be proved by induction, the second (Difference-of-powers formula) by direct calculation, using summations.

$$\text{For all real numbers } a \text{ and } b, \text{ and for all } n \in \mathbb{N}, \quad (a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r,$$

$$\text{where } \binom{n}{r} := \frac{n!}{r!(n-r)!} \text{ and where } 0! := 1 \text{ and } n! := n \cdot (n-1)! \text{ for all } n \in \mathbb{N} \text{ such that } n > 1.$$

$$\text{For all real numbers } a \text{ and } b, \text{ and for all } n \in \mathbb{N}, \quad a^n - b^n = \sum_{r=0}^{n-1} a^{n-1-r} b^r.$$

Both of these formulas are used in the proof of the existence of square roots. The proof here can then be slightly changed to create the proof that n-th roots of non-negative real numbers exist, for all $n \in \mathbb{N}$.

The square-root theorem

Theorem: For all real y, if $y \geq 0$ then there exists one and only one non-negative number x such that $x^2 = y$.

Proof: First we will prove existence, then uniqueness. In an existence proof we usually expect to use LUB, or some Theorem that uses LUB in its proof. **Notation:** We denote our $x \geq 0$ that solves $x^2 = y$ by $\sqrt{y}$

There is a vey easy case: $y = 0$. If $y = 0$ we can choose $x = 0$, and we find that $x^2 = 0$, so 0 is its own square root. Why is 0 the *only* solutionm of $x^2 = 0$? (Note! A Theorem was applied in the last sentence! Which one?)

To continue, we may suppose that $y > 0$. Let us use this set: $S := \{t \in \mathbb{R} : t \geq 0 \text{ and } t^2 < y\}$. The idea is to show that $\sigma = \sup S$ exists, and then prove that $\sigma^2 = y$.

Proof that $\sup S$ exists

First, we can show that $S \neq \emptyset$ by guessing a number that is in S. Here is a guess (that depends on the Binomial Theorem): $t := y/(y + \frac{1}{2})$. For $t^2 = \frac{y^2}{y^2+y+\frac{1}{4}} < \frac{y^2}{y} = y$ (the inequality is obtained by dropping $y^2 + \frac{1}{4}$ from the denominator, which action produces a larger fraction. (Note: If we replace 2 by n, we use $t := y/(y + \frac{1}{n})$.) Thus S is not empty.

Next, we need to show that S is bounded above. Let's show that $\frac{y}{2} + 1$ is an upper bound for S. Instead of showing that $t \in S \Rightarrow t \leq \frac{y}{2} + 1$, let us show the contrapositive, namely $t > \frac{y}{2} + 1 \Rightarrow t \notin S$. We see that $t > 0$, so, to show that these $t \notin S$ we must show that $t^2 \geq y$. Now

$$t^2 > (\frac{y}{2} + 1)^2 = \frac{y^2}{4} + y + 1 > y,$$

so t cannot belong to S. The contraposivive of the contrapositive is the original statement, so we have proved that S is bounded above. (Note: For n-th roots we use $\frac{y}{n} + 1$)

Then, by Axiom LUB, S has a least upper bound, σ, and we write $\sup S := \sigma$ as a short way to express this.

Proof that $\sigma^2 = y$

If we can prove that the statements $\sigma^2 > y$ and $\sigma^2 < y$ are false, then by the trichotomy part of the Order Axioms, $\sigma^2 = y$.

Proof that $\sigma^2 > y$ is false

Suppose that $\sigma^2 > y$. The idea is, by subtracting a positive number h_1 from σ, to construct a number $\sigma - h_1 < \sigma$ such that NO member of S is larger than it. This will contradict the part of LUB that says there exists a member of S larger than $\sigma - h_1$.

Let us set $\epsilon := \sigma^2 - y > 0$. We start with $0 < h < \sigma$, and use the Difference-of-powers formula:

$$\sigma^2 - (\sigma - h)^2 = (\sigma - (\sigma - h))(\sigma + (\sigma - h)) = h(2\sigma - h) < h(2\sigma).$$

We can now pick h_1. Our first try is: $h_1 := \epsilon/2\sigma$. But, in case $\epsilon/2\sigma \geq \sigma$, our h_1 will be too large. In this exceptional case, we can (arbitrarily) choose $h_1 = \sigma/2$ (actually, this second possibility cannot occur, but it would take extra work to check that). No matter which of these two ways we choose h_1, $h_1(2\sigma) \leq \epsilon$. Hence, if $0 < h < h_1$,

$$\sigma^2 - (\sigma - h)^2 < h(2\sigma) < h_1(2\sigma) < \epsilon = \sigma^2 - y.$$

Therefore, if $0 < h < h_1$, then $(\sigma - h)^2 > y$, so $\sigma - h \notin S$. Thus, if $\sigma - h_1 < t < \sigma$, and we define $h := \sigma - t$, then

$$0 < h = \sigma - t < \sigma - (\sigma - h_1) = h_1, \quad \text{and} \quad t = \sigma - h, \quad \text{so} \quad t \notin S.$$

So what? Well, this contradicts something we know to be true of σ. To see why, we let $\tau := \sigma - h_1$.

We know: if $\tau < \sigma$ then there exists $t \in S$ such that $\tau < t$ (something known about σ).

But we have shown that for *every* t such that $\sigma - h_1 < t < \sigma$, it is true that $t \notin S$. Note: For those with a fine sense for detail, we can show that (in this case) $\sigma \notin S$, and if $t > \sigma$ then $t \notin S$ either; we can then simply say that $\sigma - h_1$ is an upper bound for S that is less than the least upper bound.

Thus we have proved that $\sigma^2 > y$ is false. Note: For n-th roots, we'd use the Difference-of-powers formula and get

$$\sigma^n - (\sigma - h)^n = (\sigma - (\sigma - h)) \sum_{r=0}^{n-1} \sigma^{n-1-r}(\sigma - h)^r < (\sigma - (\sigma - h)) \sum_{r=0}^{n-1} \sigma^{n-1-r}(\sigma)^r = h(n\sigma^{n-1})$$

and continue in a similar manner.

Proof that $\sigma^2 < y$ is false

We know that $0 < y/(y + \frac{1}{2}) \leq \sigma$, so $\sigma > 0$, meaning that $\sigma \neq 0$. Therefore, assuming that $\sigma^2 < y$ means that $\sigma \in S$. The idea is to show that we can find $h > 0$ so that $\sigma + h \in S$, giving an element of S that is larger than the upper bound σ. This will give us the contradiction that we need.

We set $\epsilon := y - \sigma^2 > 0$. Then by the Difference-of-powers formula, for all $h > 0$,

$$(\sigma + h)^2 - \sigma^2 = h((\sigma + h) + \sigma) < h(2(\sigma + h)).$$

We'll next impose a cap on how big h can be. We'll be free to make h be even smaller if we need to. So let's assume $0 < h \leq 1$. Then

$$(\sigma + h)^2 - \sigma^2 < 2(\sigma + 1)h.$$

We are free to replace one h by 1 and leave the othjer one alone! Next, let $h := \min\left(1, \dfrac{\epsilon}{2(\sigma + 1)}\right)$. Then

$$(\sigma + h)^2 - \sigma^2 < \epsilon = y - \sigma^2, \quad \text{so} \quad (\sigma + h)^2 < y.$$

This means $\sigma + h \in S$, and gives the desired contradiction. Note: For n-th roots, we'd use the use the Difference-of-powers formula and the assumption $0 < h \leq 1$ to get

$$(\sigma + h)^n - \sigma^n = h \sum_{r=0}^{n-1} (\sigma + h)^{n-1-r} \sigma^r < h \sum_{r=0}^{n-1} (\sigma + 1)^{n-1} = h(n(\sigma + 1)^{n-1}), \quad \text{and then proceed similarly.}$$

This completes the proof of existence.

Proof of uniqueness: If $x \geq 0$ and $\sigma \geq 0$ and $x^2 = \sigma^2 = y \geq 0$, then $0 = (x - \sigma)(x + \sigma)$. Hence at least one of the factors is zero. If $x > 0$ or $\sigma > 0$, then $x + \sigma > 0$, so $x = \sigma$. If $x = 0 = \sigma$ then $x = \sigma$.

This completes the proof of the Theorem.