Problem 5

We will use the Chinese remainder theorem.

Let $\{\mathfrak{m}_i\}_{i=1}^n$ be the set of all the maximal ideals of R. By the Corollary to the Chinese remainder theorem we have

$$R/\Big(\bigcap_{i=1}^n \mathfrak{m}_i\Big) \approx R/\mathfrak{m}_1 \oplus \cdots \oplus R/\mathfrak{m}_n$$

by the canonical mapping.

It is relatively easy to see that both sides only have a finite number of invertible elements. For, since each term in the direct sum on the right hand side is a field, the invertible elements are the elements that are non-zero in every coordinate. Thus, each invertible element on the right hand side corresponds to an element $x \in R$ that is not in any of the $\mathfrak{m}_i$. However, if x is not contained in any of the maximal ideals x must be invertible, and there are only a finite number of invertible elements in R.

The set of invertible elements on the right hand side is $(R/\mathfrak{m}_1)^* \oplus \cdots \oplus (R/\mathfrak{m}_n)^*$, and since this is a finite set, we must have each set $(R/\mathfrak{m}_i)^*$ being a finite set. Also, since $R/\mathfrak{m}_i$ is a field the only non-unit is the zero element. Thus $R/\mathfrak{m}_i$ is a finite set for every i, and so the direct sum on the right hand side is finite.

Also, we can see that the set $\bigcap_{i=1}^n \mathfrak{m}_i$ is finite. For, if $x \in \mathfrak{m}_i$, then $1 + x \notin \mathfrak{m}_i$. This is true for every i, so $1 + x$ is not in any of the maximal ideals. Thus $x + 1$ is invertible. Thus we have an injective mapping from $\bigcap_{i=1}^n \mathfrak{m}_i$ to R^*, which is a finite set.

Thus, since

$$|R|/\left|\Big(\bigcap_{i=1}^n \mathfrak{m}_i\Big)\right| = \left|R/\Big(\bigcap_{i=1}^n \mathfrak{m}_i\Big)\right| = |R/\mathfrak{m}_1 \oplus \cdots \oplus R/\mathfrak{m}_n|,$$

and since $\left|\Big(\bigcap_{i=1}^n \mathfrak{m}_i\Big)\right|$ and $|R/\mathfrak{m}_1 \oplus \cdots \oplus R/\mathfrak{m}_n|$ are both finite, we must have $|R|$ finite as well.