Quenched Limits for Transient, Ballistic, Sub-Gaussian One-Dimensional Random Walk in Random Environment

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Abstract

We consider a nearest-neighbor, one-dimensional random walk $\{X_n\}_{n\geq 0}$ in a random i.i.d. environment, in the regime where the walk is transient with speed $v_P > 0$ and there exists an $s \in (1,2)$ such that the annealed law of $n^{-1/s}(X_n - nv_P)$ converges to a stable law of parameter s. Under the quenched law (i.e., conditioned on the environment), we show that no limit laws are possible. In particular we show that there exist sequences $\{t_k\}$ and $\{t'_k\}$ depending on the environment only, such that a quenched central limit theorem holds along the subsequence t_k, but the quenched limiting distribution along the subsequence t'_k is a centered reverse exponential distribution. This complements the results of a recent paper of Peterson and Zeitouni (*arXiv:math/0704.1778v1 [math.PR]*) which handled the case when the parameter $s \in (0,1)$.

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1 Introduction, Notation, and Statement of Main Results

Let $\Omega = [0,1]^{\mathbb{Z}}$, and let $\mathcal{F}$ be the Borel $\sigma-$algebra on Ω. A random environment is an Ω-valued random variable $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ with distribution P. In this paper we will assume that P is a product measure on Ω. The *quenched* law P_ω^x for a random walk X_n in the environment ω is defined by

$$P_\omega^x(X_0 = x) = 1, \quad \text{and} \quad P_\omega^x\left(X_{n+1} = j | X_n = i\right) = \begin{cases} \omega_i & \text{if } j = i+1, \\ 1-\omega_i & \text{if } j = i-1. \end{cases}$$

$\mathbb{Z}^{\mathbb{N}}$ is the space for the paths of the random walk $\{X_n\}_{n\in\mathbb{N}}$, and let $\mathcal{G}$ denote the $\sigma-$algebra generated by the cylinder sets. Note that for each $\omega \in \Omega$, P_ω is a probability measure on $(\mathbb{Z}^{\mathbb{N}}, \mathcal{G})$, and for each $G \in \mathcal{G}$, $P_\omega^x(G) : (\Omega, \mathcal{F}) \to [0,1]$ is a measurable function of ω. Expectations under

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the law P_ω^x are denoted E_ω^x. The *annealed* law for the random walk in random environment X_n is defined by

$$\mathbb{P}^x(F \times G) = \int_F P_\omega^x(G) P(d\omega), \quad F \in \mathcal{F}, G \in \mathcal{G}.$$

For ease of notation we will use P_ω and $\mathbb{P}$ in place of P_ω^0 and $\mathbb{P}^0$ respectively. We will also use $\mathbb{P}^x$ to refer to the marginal on the space of paths, i.e. $\mathbb{P}^x(G) = \mathbb{P}^x(\Omega \times G) = E_P\left[P_\omega^x(G)\right]$ for $G \in \mathcal{G}$. Expectations under the law $\mathbb{P}$ will be written $\mathbb{E}$.

A simple criterion for recurrence of a one-dimensional RWRE and a formula for the speed of transience was given by Solomon in [11]. For any integers $i \le j$ define

$$\rho_i := \frac{1-\omega_i}{\omega_i}, \quad \text{and} \quad \Pi_{i,j} := \prod_{k=i}^{j} \rho_k \,. \tag{1}$$

Then, X_n is transient to the right (resp. to the left) if $E_P(\log \rho_0) < 0$, (resp. $E_P \log \rho_0 > 0$) and recurrent if $E_P(\log \rho_0) = 0$ (henceforth we will write ρ instead of ρ_0 in expectations involving only ρ_0). In the case where $E_P \log \rho < 0$ (transience to the right), Solomon established the following law of large numbers

$$v_P := \lim_{n\to\infty} \frac{X_n}{n} = \lim_{n\to\infty} \frac{n}{T_n} = \frac{1}{\mathbb{E}T_1}, \quad \mathbb{P}-a.s. \tag{2}$$

where $T_n := \min\{k \ge 0 : X_k = n\}$. For any integers $i < j$ define

$$W_{i,j} := \sum_{k=i}^{j} \Pi_{k,j}, \quad \text{and} \quad W_j := \sum_{k \le j} \Pi_{k,j} \,. \tag{3}$$

When $E_P \log \rho < 0$, it was shown in [12] that

$$E_\omega^j T_{j+1} = 1 + 2W_j < \infty, \quad P-a.s., \tag{4}$$

and thus $v_P = 1/(1+2E_P W_0)$. Since P is a product measure, $E_P W_0 = \sum_{k=1}^{\infty} \left(E_P \rho\right)^k$. In particular, $v_P > 0$ if $E_P \rho < 1$.

Kesten, Kozlov, and Spitzer [6] determined the annealed limiting distribution of a RWRE with $E_P \log \rho < 0$, i.e. transient to the right. They derived the limiting distributions for the walk by first establishing a stable limit law of index s for T_n, where s is defined by the equation $E_P \rho^s = 1$. In particular, they showed that when $s \in (1,2)$ there exists a $b > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_n - \mathbb{E}T_n}{n^{1/s}} \le x \right) = L_{s,b}(x) \tag{5}$$

and

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_n - n v_P}{v_P^{1+1/s} n^{1/s}} \le x \right) = 1 - L_{s,b}(-x), \tag{6}$$

where $L_{s,b}$ is the distribution function for a stable random variable with characteristic function

$$\hat{L}_{s,b}(t) = \exp\left\{ -b|t|^s \left(1 - i\frac{t}{|t|} \tan(\pi s/2) \right) \right\}.$$

While the annealed limiting distributions for transient one-dimensional RWRE have been known for quite a while, the corresponding quenched limiting distributions have remained largely unstudied until recently. In the case when $s > 2$, Goldsheid [4] and Peterson [10] independently proved that a quenched CLT holds with a random (depending on the environment) centering. Previously, in [8] and [12] it had only been shown that the limiting statements for the quenched CLT with random centering held in probability (rather than almost surely). In the case when $s < 1$ it was shown in [9] that no quenched limiting distribution exists for the RWRE. In particular, it was shown that $P-a.s.$ there exist two different random sequences t_k and t'_k such that the behavior of the RWRE is either localized (concentrated in a interval of size $\log^2 t'_k$) or spread out (scaling of order t_k^s).

In this paper, we analyze the quenched limiting distributions of a one-dimensional transient RWRE in the case $s \in (1,2)$. We show that, as in the case when $s < 1$, there is no quenched limiting distribution of the random walk. However, as shown in Section 2, the existence of a positive speed for the random walk allows us to transfer limiting distributions from T_n to X_n. Throughout the paper, we will make the following assumptions:

Assumption 1. *P is a product measure on Ω such that*

$$E_P \log\rho < 0 \quad \textit{and} \quad E_P\rho^s = 1 \textit{ for some } s > 0. \tag{7}$$

Assumption 2. *The distribution of $\log\rho$ is non-lattice under P and $E_P(\rho^s\log\rho) < \infty$.*

Remarks:
1. Assumption 1 contains the essential assumptions for our results. The technical conditions contained in Assumption 2 were also invoked in [6] and [9].
2. Since $E_P\rho^\gamma$ is a convex function of γ, the two statements in (7) give that $E_P\rho^\gamma < 1$ for all $0 < \gamma < s$ and $E_P\rho^\gamma > 1$ for all $\gamma > s$. In particular this implies that $v_P > 0 \iff s > 1$. The main results of this paper are for $s \in (1,2)$, but many statements hold for a wider range of s. If no mention is made of bounds on s then it is assumed that the statement holds for all $s > 0$.
3. The cases $s \in 1,2$ are not covered by [9] or by this paper. It is not clear whether or not a quenched CLT holds in the case $s = 2$, but we suspect that the results for $s = 1$ will be similar to those of the cases $s \in (0,1)$ and $s \in (1,2)$ - i.e. no quenched limiting distribution for the random walk. However, since $s = 1$ is the bordering case between the zero-speed and positive-speed regimes the analysis is likely to be more technical (as was also the case in [6]).

Let $\Phi(x)$ and $\Psi(x)$ be the distribution functions for a gaussian and exponential random variable respectively. That is,

$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}dt \quad \text{and} \quad \Psi(x) := \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}.$$

Our main results are the following:

Theorem 1.1. *Let Assumptions 1 and 2 hold, and let $s \in (1,2)$. Then $P-a.s.$ there exists a random subsequence $n_{k_m} = n_{k_m}(\omega)$ of $n_k = 2^{2^k}$ and non-deterministic random variables $v_{k_m,\omega}$ such that*

$$\lim_{m\to\infty} P_\omega\left(\frac{T_{n_{k_m}} - E_\omega T_{n_{k_m}}}{\sqrt{v_{k_m,\omega}}} \leq x \right) = \Phi(x), \qquad \forall x \in \mathbb{R},$$

and

$$\lim_{m\to\infty} P_\omega\left(\frac{X_{t_m} - n_{k_m}}{v_P\sqrt{v_{k_m,\omega}}} \leq x \right) = \Phi(x), \qquad \forall x \in \mathbb{R},$$

where $t_m = t_m(\omega) := \lfloor E_\omega T_{n_{k_m}} \rfloor$.

Theorem 1.2. *Let Assumptions 1 and 2 hold, and let* $s \in (1,2)$. *Then* $P - a.s.$ *there exists a random subsequence* $n_{k_m} = n_{k_m}(\omega)$ *of* $n_k = 2^{2^k}$ *and non-deterministic random variables* $v_{k_m,\omega}$ *such that*

$$\lim_{m\to\infty} P_\omega \left(\frac{T_{n_{k_m}} - E_\omega T_{n_{k_m}}}{\sqrt{v_{k_m,\omega}}} \leq x \right) = \Psi(x+1), \qquad \forall x \in \mathbb{R},$$

and

$$\lim_{m\to\infty} P_\omega \left(\frac{X_{t_m} - n_{k_m}}{v_P \sqrt{v_{k_m,\omega}}} \leq x \right) = 1 - \Psi(-x+1), \qquad \forall x \in \mathbb{R},$$

where $t_m = t_m(\omega) := \lfloor E_\omega T_{n_{k_m}} \rfloor$.

Remarks:
1. Note that Theorems 1.1 and 1.2 preclude the possiblity of quenched analogues of the annealed statements (5) and (6).
2. The sequence $n_k = 2^{2^k}$ in Theorems 1.1 and 1.2 is chosen only for convenience. In fact, for any sequence n_k growing sufficiently fast, $P - a.s.$ there will be a random subsequence $n_{k_m}(\omega)$ such that the conclusions of Theorems 1.1 and 1.2 hold.
3. The definition of $v_{k_m,\omega}$ is given below in (10), and similar to Theorem 1.3, it can be shown that $\lim_{n\to\infty} P\left(n_k^{-2/s} v_{k,\omega} \leq x \right) = L_{\frac{s}{2},b}(x)$ for some $b > 0$. Also, from (2) we have that $t_m \sim \mathbb{E}T_1 n_{k_m}$. Thus, the scaling in Theorems 1.1 and 1.2 is of the same order as the annealed scaling but cannot be replaced by a deterministic scaling.

As in [9], define the "ladder locations" ν_i of the environment by

$$\nu_0 = 0, \quad \text{and} \quad \nu_i = \begin{cases} \inf\{n > \nu_{i-1} : \Pi_{\nu_{i-1},n-1} < 1\}, & i \geq 1, \\ \sup\{j < \nu_{i+1} : \Pi_{k,j-1} < 1, \quad \forall k < j\}, & i \leq -1\,. \end{cases} \tag{8}$$

Throughout the remainder of the paper we will let $\nu = \nu_1$. We will sometimes refer to sections of the environment between ν_{i-1} and $\nu_i - 1$ as "blocks" of the environment. Note that the block between ν_{-1} and $\nu_0 - 1$ is different from all the other blocks between consecutive ladder locations (in particular it can be that $\Pi_{\nu_{-1},\nu_0-1} \geq 1$), and that all the other blocks have the same distribution as the block from 0 to $\nu - 1$. As in [9] we define the measure Q on environments by $Q(\cdot) := P(\cdot|\mathcal{R})$, where

$$\mathcal{R} := \{\omega \in \Omega : \Pi_{-k,-1} < 1, \quad \forall k \geq 1\} = \left\{ \omega \in \Omega : \sum_{i=-k}^{-1} \log \rho_i < 0, \quad \forall k \geq 1 \right\}.$$

Note that $P(\mathcal{R}) > 0$ since $E_P \log \rho < 0$. Q is defined so that the blocks of the environment between ladder locations are i.i.d. under Q, all with distribution the same as that of the block from 0 to $\nu - 1$ under P. In particular P and Q agree on $\sigma(\omega_i : i \geq 0)$.

For any random variable Z, define the quenched variance $Var_\omega Z := E_\omega(Z - E_\omega Z)^2$. In [9, Theorem 1.1] it was proved that when $s \in (0,1)$, $n^{-1/s} E_\omega T_{\nu_n}$ converges in distribution (under Q) to a stable distribution of index s. Correspondingly, when $s < 2$ we will prove the following theorem:

Theorem 1.3. *Let Assumptions 1 and 2 hold, and let* $s < 2$. *Then there exists a* $b > 0$ *such that*

$$\lim_{n\to\infty} Q\left(\frac{Var_\omega T_{\nu_n}}{n^{2/s}} \leq x \right) = \lim_{n\to\infty} Q\left(\frac{1}{n^{2/s}} \sum_{i=1}^{n} \left(E_\omega^{\nu_{i-1}} T_{\nu_i} \right)^2 \leq x \right) = L_{\frac{s}{2},b}(x)\,. \tag{9}$$

Remarks:
1. The constant b in the above theorem may not be the same as in (5) and (6).
2. Theorem 1.3 can be used to show that $\lim_{n\to\infty} P\left(\frac{Var_\omega T_n}{n^{2/s}} \leq x\right) = L_{\frac{s}{2},b'}(x)$ for some $b' > 0$, but we will not prove this since we do not use it for the other results in this paper.

A major difficulty in analyzing T_{ν_n} is that the crossing time from ν_{i-1} to ν_i depends on the entire environment to the left of ν_i. Thus $Var_\omega(T_{\nu_i} - T_{\nu_{i-1}})$ and $Var_\omega(T_{\nu_j} - T_{\nu_{j-1}})$ are not independent even if $|i-j|$ is large. However, since the RWRE will generally not backtrack too far, the dependence on the environment far to the left is quite weak (indeed [3, Lemma 3.3] gives that $\mathbb{P}(T_{-k} < \infty) \leq e^{-Ck}$ for some $C > 0$). Thus, with minimal probabilistic cost we can add reflections to the RWRE to make crossing times of blocks that are far apart independent. For $n = 1, 2, \ldots$, define $b_n := \lfloor \log^2(n) \rfloor$. Then, as in [9] let $\bar{X}_t^{(n)}$ be the random walk that is the same as X_t with the added condition that after reaching ν_k the environment is modified by setting $\omega_{\nu_{k-b_n}} = 1$ (i.e. never allow the walk to backtrack more than $\log^2(n)$ blocks). Denote by $\bar{T}_x^{(n)}$ the corresponding hitting times. For ease of notation we define

$$\mu_{i,n,\omega} := E_\omega^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)}, \quad \text{and} \quad \sigma_{i,n,\omega}^2 := Var_\omega\left(\bar{T}_{\nu_i}^{(n)} - \bar{T}_{\nu_{i-1}}^{(n)}\right).$$

The structure of the paper is as follows: In Section 2 we prove the following general proposition that allows us to easily transfer quenched limit laws from subsequences of T_n to X_n.

Proposition 1.4. *Let Assumptions 1 and 2 hold, and let $s \in (1,2)$. Also, let n_k be a sequence of integers growing fast enough so that $\lim_{k\to\infty} \frac{n_k}{n_{k-1}^{1+\delta}} = \infty$ for some $\delta > 0$, and define*

$$d_k := n_k - n_{k-1}, \quad \textit{and} \quad v_{k,\omega} := \sum_{i=n_{k-1}+1}^{n_k} \sigma_{i,d_k,\omega}^2 = Var_\omega\left(\bar{T}_{\nu_{n_k}}^{(d_k)} - \bar{T}_{\nu_{n_{k-1}}}^{(d_k)}\right). \tag{10}$$

Assume that F is a continuous distribution function for which $P-a.s.$ there exists a subsequence $n_{k_m} = n_{k_m}(\omega)$ such that for $\alpha_m := n_{k_m - 1}$,

$$\lim_{m\to\infty} P_\omega^{\nu_{\alpha_m}}\left(\frac{\bar{T}_{x_m}^{(d_{k_m})} - E_\omega^{\nu_{\alpha_m}} \bar{T}_{x_m}^{(d_{k_m})}}{\sqrt{v_{k_m,\omega}}} \leq y\right) = F(y), \quad \forall y \in \mathbb{R},$$

for any sequence $x_m \sim n_{k_m}$. Then, $P-a.s.$ for all $y \in \mathbb{R}$ we also have

$$\lim_{m\to\infty} P_\omega\left(\frac{T_{x_m} - E_\omega T_{x_m}}{\sqrt{v_{k_m,\omega}}} \leq y\right) = F(y), \tag{11}$$

for any $x_m \sim n_{k_m}$, and

$$\lim_{m\to\infty} P_\omega\left(\frac{X_{t_m} - n_{k_m}}{v_P\sqrt{v_{k_m,\omega}}} \leq y\right) = 1 - F(-y), \tag{12}$$

where $t_m := \left\lfloor E_\omega T_{n_{k_m}} \right\rfloor$.

Then in Sections 3 and 4 we use Theorem 1.3 to find subsequences $n_{k_m}(\omega)$ that allow us to apply Proposition 1.4. To find a subsequence that gives Gaussian behavior of $T_{n_{k_m}}$ we find a subsequence where none of the crossing times of the first n_{k_m} blocks is too much larger than all the others and then use the Linberg-Feller condition for triangular arrays. In contrast, to find a subsequence that gives exponential behavior of $T_{n_{k_m}}$ we first prove that the crossing times of "large" blocks is approximately exponential in distribution. Then we find a subsequence where the crossing time of one of the first n_{k_m} blocks dominates the total crossing time of the first n_{k_m} blocks. Finally, Section 5 contains the proof of Theorem 1.3 which is similar to that of [9, Theorem 1.1].

Before continuing with the proofs of the main theorems we recall some notation and results from [9] that will be used throughout the paper. First, recall that from [9, Lemma 2.1] there exist constants $C_1, C_2 > 0$ such that

$$P(\nu > x) \leq C_1 e^{-C_2 x}, \quad \forall x \geq 0. \tag{13}$$

Then, since $\nu_n = \sum_{i=1}^n \nu_i - \nu_{i-1}$ and the $\nu_i - \nu_{i-1}$ are i.i.d., the law of large numbers gives that

$$\lim_{n\to\infty} \frac{\nu_n}{n} = E_P \nu =: \bar{\nu} < \infty, \qquad P-a.s. \tag{14}$$

In [9] the following formulas for the quenched expectation and variance of T_ν were given:

$$E_\omega T_\nu = \nu + 2\sum_{j=0}^{\nu-1} W_j, \quad \text{and} \quad Var_\omega T_\nu = 4\sum_{j=0}^{\nu-1}(W_j + W_j^2) + 8\sum_{j=0}^{\nu-1}\sum_{i<j} \Pi_{i+1,j}(W_i + W_i^2), \tag{15}$$

Note that since the added reflections only decrease crossing times we obviously have $T_\nu \geq \bar{T}_\nu^{(n)}$ and $E_\omega T_\nu \geq E_\omega \bar{T}_\nu^{(n)}$ for any n. Also, since (15) holds for any environment ω, the formula for $Var_\omega \bar{T}_\nu^{(n)}$ is the same as in (15) but with $\rho_{\nu_{-b_n}}$ replaced by 0. In particular, this shows that $Var_\omega T_\nu \geq Var_\omega \bar{T}_\nu^{(n)}$ for any n. As in [9] define for any integer i

$$M_i := \max\{\Pi_{\nu_{i-1},j} : j \in [\nu_{i-1}, \nu_i)\}\,. \tag{16}$$

Then [5, Theorem 1] gives that there exists a constant $C_3 < \infty$ such that

$$Q(M_i > x) = P(M_1 > x) \sim C_3 x^{-s}. \tag{17}$$

Note that $M_1 \leq \max_{0\leq j<\nu} W_j$. Therefore, from the formulas for $E_\omega T_\nu$ and $Var_\omega T_\nu$ in (15) it is easy to see that $E_\omega T_\nu \geq M_1$ and $Var_\omega T_\nu \geq M_1^2$ (the same also being true with $\bar{T}_\nu^{(n)}$). Finally, recall the following results from [9]:

Theorem 1.5 (Lemma 3.3 & Theorem 5.1 in [9]). *There exists a constant $K_\infty \in (0,\infty)$ such that*

$$Q\left(Var_\omega T_\nu > x\right) \sim Q\left((E_\omega T_\nu)^2 > x\right) \sim K_\infty x^{-s/2}, \qquad \text{as } x \to \infty.$$

Moreover, for any $\varepsilon > 0$ and $x > 0$

$$Q\left(Var_\omega \bar{T}_\nu^{(n)} > xn^{2/s},\, M_1 > n^{(1-\varepsilon)/s}\right) \sim Q\left(\left(E_\omega \bar{T}_\nu^{(n)}\right)^2 > xn^{2/s},\, M_1 > n^{(1-\varepsilon)/s}\right) \sim K_\infty x^{-s/2}\frac{1}{n},$$

as $n \to \infty$.

2 Converting Time Limits to Space Limits

In this section we develop a general method for transferring a quenched limit law for a subsequence of T_n to a quenched limit law for a subsequence of X_n. We begin with some lemmas analyzing the a.s. asymptotic behavior of the quenched variance and mean of the hitting times.

Lemma 2.1. *Assume $s \leq 2$. Then for any $\delta > 0$,*

$$Q\left(Var_\omega \bar{T}^{(n)}_{\nu_n} \notin \left(n^{2/s-\delta}, n^{2/s+\delta}\right)\right) \leq \frac{1}{P(\mathcal{R})} P\left(Var_\omega \bar{T}^{(n)}_{\nu_n} \notin \left(n^{2/s-\delta}, n^{2/s+\delta}\right)\right) = o\left(n^{-\delta s/4}\right).$$

Proof. The first inequality in the lemma is trivial since for any $A \in \mathcal{F}$ we have from the definition of Q that $Q(A) = \frac{P(A\cap\mathcal{R})}{P(\mathcal{R})} \leq \frac{P(A)}{P(\mathcal{R})}$. Next, note that when $s \leq 2$ [9, Lemma 5.11] gives

$$P\left(Var_\omega \bar{T}^{(n)}_{\nu_n} \geq n^{2/s+\delta}\right) \leq P\left(Var_\omega T_{\nu_n} \geq n^{2/s+\delta}\right) = o(n^{-\delta s/4}). \tag{18}$$

Also, since $Var_\omega(\bar{T}^{(n)}_{\nu_i} - \bar{T}^{(n)}_{\nu_{i-1}}) \geq M_i^2$ we have

$$P\left(Var_\omega \bar{T}^{(n)}_{\nu_n} \leq n^{2/s-\delta}\right) \leq P\left(M_1^2 \leq n^{2/s-\delta}\right)^n = \left(1 - P\left(M_1 > n^{1/s-\delta/2}\right)\right)^n = o\left(e^{-n^{\delta s/4}}\right),$$

where the last equality is from (17). □

Corollary 2.2. *Assume $s \leq 2$. Then for any $\delta > 0$*

$$P\left(v_{k,\omega} \notin \left(d_k^{2/s-\delta}, d_k^{2/s+\delta}\right)\right) = o\left(d_k^{-\delta s/4}\right).$$

Consequently, if $s < 2$ we have $\sqrt{v_{k,\omega}} = o(d_k)$, $P-a.s.$

Proof. Recall from (10) that by definition $v_{k,\omega} = Var_\omega\left(\bar{T}^{(d_k)}_{\nu_{n_k}} - \bar{T}^{(d_k)}_{\nu_{n_{k-1}}}\right)$. Also, note that the conditions on n_k ensure that n_k grows faster than exponentially and that $d_k \sim n_k$. Thus, for all k large enough $v_{k,\omega}$ only depends on the environment to the right of zero. Therefore for all k large enough

$$\begin{aligned}P\left(v_{k,\omega} \notin \left(d_k^{2/s-\delta}, d_k^{2/s+\delta}\right)\right) &= Q\left(Var_\omega\left(\bar{T}^{(d_k)}_{\nu_{n_k}} - \bar{T}^{(d_k)}_{\nu_{n_{k-1}}}\right) \notin \left(d_k^{2/s-\delta}, d_k^{2/s+\delta}\right)\right)\\ &= Q\left(Var_\omega \bar{T}^{(d_k)}_{\nu_{d_k}} \notin \left(d_k^{2/s-\delta}, d_k^{2/s+\delta}\right)\right) = o\left(d_k^{-\delta s/4}\right),\end{aligned}$$

where the last equality is from Lemma 2.1. Now, for the second claim in the corollary, first note that $2 > \frac{2}{s} + \frac{s-1}{s}$ since $s > 1$. Therefore, for any $\varepsilon > 0$ and for all k large enough we have

$$P\left(v_{k,\omega} > \varepsilon d_k^2\right) \leq P\left(v_{k,\omega} > d_k^{2/s+(s-1)/s}\right) = o\left(d_k^{-(s-1)/4}\right).$$

This last term is summable since d_k grows faster than exponentially. Thus the Borel-Cantelli Lemma gives that $v_{k,\omega} = o(d_k^2)$, $P-a.s.$ □

Corollary 2.3. *Assume $s \leq 2$. Then*

$$\lim_{k\to\infty} \frac{Var_\omega T_{\nu_{n_{k-1}}}}{v_{k,\omega}} = 0, \quad P-a.s.$$

Proof. By the Borel-Cantelli Lemma it is enough to prove that for any $\varepsilon > 0$

$$\sum_{k=1}^{\infty} P\left(Var_\omega T_{\nu_{n_{k-1}}} \geq \varepsilon v_{k,\omega}\right) < \infty$$

However, for any $\delta > 0$ we have

$$P\left(Var_\omega T_{\nu_{n_{k-1}}} \geq \varepsilon v_{k,\omega}\right) \leq P\left(Var_\omega T_{\nu_{n_{k-1}}} \geq \varepsilon d_k^{2/s-\delta}\right) + P\left(v_{k,\omega} \leq d_k^{2/s-\delta}\right). \tag{19}$$

By Corollary 2.2 the last term in (19) is summable for any $\delta > 0$. To show that the second to last term in (19) is also summable first note that the conditions on the sequence n_k give that there exists a $\delta > 0$ such that $\varepsilon d_k^{2/s-\delta} \geq n_{k-1}^{2/s+\delta}$ for all k large enough. Thus, for some $\delta > 0$ and all k large enough we have

$$P\left(Var_\omega T_{\nu_{n_{k-1}}} > \varepsilon d_k^{2/s+\delta}\right) \leq P\left(Var_\omega T_{\nu_{n_{k-1}}} > n_{k-1}^{2/s-\delta}\right) = o(n_{k-1}^{-\delta s/4}),$$

where the last equality is from (18). $\square$

Lemma 2.4. *Assume $s \in (1,2)$. Then $\mathbb{E}T_1 < \infty$, and $P-a.s.$*

$$\lim_{k\to\infty} \frac{E_\omega T_{n_k+\lceil x\sqrt{v_{k,\omega}}\rceil} - E_\omega T_{n_k}}{\sqrt{v_{k,\omega}}} = x\mathbb{E}T_1, \quad \forall x \in \mathbb{R}. \tag{20}$$

Proof. Now, since $\frac{E_\omega T_{n_k+\lceil x\sqrt{v_{k,\omega}}\rceil} - E_\omega T_{n_k}}{\sqrt{v_{k,\omega}}}$ is monotone in x it is enough to prove that for arbitrary $x \in \mathbb{Q}$ the limiting statement in (20) holds. Obviously this is true when $x = 0$ since both sides are zero. For the remainder of the proof we'll assume $x > 0$. The proof for $x < 0$ is essentially the same (recall that by Corollary 2.2 $v_{k,\omega} = o(d_k) = o(n_k)$ when $s < 2$). Note that for $x \geq 0$ then we can re-write $E_\omega T_{n_k+\lceil x\sqrt{v_{k,\omega}}\rceil} - E_\omega T_{n_k} = E_\omega^{n_k} T_{n_k+\lceil x\sqrt{v_{k,\omega}}\rceil}$. By the Borel-Cantelli Lemma it is enough to show that for any $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} P\left(\left|E_\omega^{n_k} T_{n_k+\lceil x\sqrt{v_{k,\omega}}\rceil} - \lceil x\sqrt{v_{k,\omega}}\rceil \mathbb{E}T_1\right| \geq \varepsilon\sqrt{v_{k,\omega}}\right) < \infty\,. \tag{21}$$

However, for any $\delta > 0$ we have

$$\begin{aligned}
&P\left(\left|E_\omega^{n_k} T_{n_k+\lceil x\sqrt{v_{k,\omega}}\rceil} - \lceil x\sqrt{v_{k,\omega}}\rceil \mathbb{E}T_1\right| \geq \varepsilon\sqrt{v_{k,\omega}}\right) \\
&\quad \leq P\left(\exists m \in \left[\lceil x d_k^{1/s-\delta}\rceil, \lceil x d_k^{1/s+\delta}\rceil\right] : |E_\omega^{n_k} T_{n_k+m} - m\mathbb{E}T_1| \geq \frac{\varepsilon m}{x}\right) + P\left(v_{k,\omega} \notin \left[d_k^{2/s-2\delta}, d_k^{2/s+2\delta}\right]\right) \\
&\quad \leq P\left(\max_{m \leq \lceil x d_k^{1/s+\delta}\rceil} |E_\omega T_m - m\mathbb{E}T_1| \geq \varepsilon d_k^{1/s-\delta}\right) + o(d_k^{-\delta s/2}),
\end{aligned} \tag{22}$$

where the last inequality is due to Corollary 2.2 and the fact that $\{E_\omega^{n_k} T_{n_k+m}\}_{m\in\mathbb{Z}}$ has the same distribution as $\{E_\omega T_m\}_{m\in\mathbb{Z}}$ since P is a product measure. Thus, we only need to show that the first term in (22) is summable in k for some $\delta > 0$. For this, we need the following lemma whose proof we defer.

Lemma 2.5. *Assume $s \in (1,2]$. Then for any $0 < \delta' < \frac{s-1}{2s}$ we have that*

$$P\left(\max_{m\leq n} |E_\omega T_m - m\mathbb{E}T_1| \geq n^{1-\delta'}\right) = o\left(n^{-(s-1)/2}\right)$$

Assuming Lemma 2.5, fix $0 < \delta' < \frac{s-1}{2s}$ and then choose $0 < \delta < \frac{\delta'}{s(2-\delta')}$. We choose δ and δ' this way to ensure that $(1/s+\delta)(1-\delta') < 1/s - \delta$. Therefore, for all k large enough, $\varepsilon d_k^{1/s-\delta} > \left\lceil x d_k^{1/s+\delta} \right\rceil^{1-\delta'}$. Thus for all k large enough we have

$$P\left(\max_{m \leq \lceil x d_k^{1/s+\delta} \rceil} |E_\omega T_m - m\mathbb{E}T_1| \geq \varepsilon d_k^{1/s-\delta} \right) \leq P\left(\max_{m \leq \lceil x d_k^{1/s+\delta} \rceil} |E_\omega T_m - m\mathbb{E}T_1| \geq \left\lceil x d_k^{1/s+\delta} \right\rceil^{1-\delta'} \right)$$
$$= o\left(d_k^{-(1/s+\delta)(s-1)/2} \right), \qquad \text{as } k \to \infty.$$

Since $s > 1$ this last term is summable in k. $\square$

Proof of Lemma 2.5: Before proceeding with the proof we need to introduce some notation for a slightly different type of reflection. Define $\tilde{X}_t^{(n)}$ to be the RWRE modified so that it cannot backtrack a distance of b_n (the definition of $\bar{X}_t^{(n)}$ is similar except the walk was not allowed to backtrack b_n blocks instead). That is, after the walk first reaches location i, we modify the environment by setting $\omega_{i-b_n} = 1$. Let $\tilde{T}_x^{(n)}$ be the corresponding hitting times of the walk $\tilde{X}_t^{(n)}$. Then

$$\begin{aligned} P\left(\max_{m \leq n} |E_\omega T_m - m\mathbb{E}T_1| \geq n^{1-\delta'} \right) &\leq P\left(E_\omega T_n - E_\omega \tilde{T}_n^{(n)} \geq \frac{n^{1-\delta'}}{3} \right) + P\left(\mathbb{E}T_1 - \mathbb{E}\tilde{T}_1^{(n)} \geq \frac{n^{-\delta'}}{3} \right) \\ &\quad + P\left(\max_{m \leq n} \left| E_\omega \tilde{T}_m^{(n)} - m\mathbb{E}\tilde{T}_1^{(n)} \right| \geq \frac{n^{1-\delta'}}{3} \right) \\ &\leq 3n^{-1+\delta'}(\mathbb{E}T_n - \mathbb{E}\tilde{T}_n^{(n)}) + \mathbf{1}_{\mathbb{E}T_1 - \mathbb{E}\tilde{T}_1^{(n)} \geq n^{-\delta'}/3} \\ &\quad + P\left(\max_{m \leq n} \left| E_\omega \tilde{T}_m^{(n)} - m\mathbb{E}\tilde{T}_1^{(n)} \right| \geq \frac{n^{1-\delta'}}{3} \right) \end{aligned} \tag{23}$$

Now, from (4) we get that $E_\omega T_1 - E_\omega \tilde{T}_1^{(n)} = (1+2W_0) - (1+2W_{-b_n+1,0}) = 2\Pi_{-b_n+1,0}W_{-b_n}$, and thus since P is a product measure

$$\mathbb{E}T_n - \mathbb{E}\tilde{T}_n^{(n)} = nE_P\left(E_\omega T_1 - E_\omega \tilde{T}_1^{(n)} \right) = \frac{2n}{1 - E_P\rho}(E_P\rho)^{b_n+1}. \tag{24}$$

Since $E_P\rho < 1$ and $b_n \sim \log^2 n$ the above decreases faster than any power of n. Thus by (23) we need only to show that $P\left(\max_{m \leq n} \left| E_\omega \tilde{T}_m^{(n)} - m\mathbb{E}\tilde{T}_1^{(n)} \right| \geq \frac{n^{1-\delta'}}{3} \right) = o(n^{-(s-1)/2})$. For ease of notation we define $\kappa_i^{(n)} := E_\omega^{i-1}\tilde{T}_i^{(n)} - \mathbb{E}\tilde{T}_1^{(n)}$. Thus, since $E_\omega \tilde{T}_m^{(n)} - m\mathbb{E}\tilde{T}_1^{(n)} = \sum_{i=1}^m \kappa_i^{(n)} = \sum_{i=1}^{b_n} \sum_{j=0}^{\left\lfloor \frac{m-i}{b_n} \right\rfloor} \kappa_{jb_n+i}^{(n)}$

we have

$$
\begin{aligned}
P\left(\max_{m\le n}\left|E_\omega \tilde{T}^{(n)}_m - m\mathbb{E}\tilde{T}^{(n)}_1\right| \ge \frac{n^{1-\delta'}}{3}\right) &\le P\left(\max_{m\le n}\sum_{i=1}^{b_n}\left|\sum_{j=0}^{\left\lfloor\frac{m-i}{b_n}\right\rfloor}\kappa^{(n)}_{jb_n+i}\right| \ge \frac{n^{1-\delta'}}{3}\right)\\
&\le \sum_{i=1}^{b_n} P\left(\max_{m\le n}\left|\sum_{j=0}^{\left\lfloor\frac{m-i}{b_n}\right\rfloor}\kappa^{(n)}_{jb_n+i}\right| \ge \frac{n^{1-\delta'}}{3b_n}\right)\\
&= \sum_{i=1}^{b_n} P\left(\max_{l\le\left\lfloor\frac{n-i}{b_n}\right\rfloor}\left|\sum_{j=0}^{l}\kappa^{(n)}_{jb_n+i}\right| \ge \frac{n^{1-\delta'}}{3b_n}\right). \qquad (25)
\end{aligned}
$$

Due to the reflections of the random walk, $\kappa^{(n)}_i$ depends only on the environment between $i-b_n$ and $i-1$. Thus, for each i $\{\kappa^{(n)}_{jb_n+i}\}_{j=0}^\infty$ is a sequence of $i.i.d.$ random variables with zero mean, and so $\{\sum_{j=0}^l \kappa^{(n)}_{jb_n+i}\}_{l\ge 0}$ is a martingale. Now, let $\gamma\in(1,s)$. Then, by the Doob-Kolmogorov inequality, for any integer N we have

$$
P\left(\max_{l\le N}\left|\sum_{j=0}^{l}\kappa^{(n)}_{jb_n+i}\right| \ge \frac{n^{1-\delta'}}{3b_n}\right) \le 3^\gamma b_n^\gamma n^{-\gamma+\gamma\delta'} E_P\left|\sum_{j=0}^{N}\kappa^{(n)}_{jb_n+i}\right|^\gamma .
$$

Now, since $\{\kappa^{(n)}_{jb_n+i}\}_{j=0}^\infty$ is a sequence of independent, zero-mean random variables, the Marcinkiewicz-Zygmund inequality [1, Theorem 2 on p. 356] gives that there exists a constant $B_\gamma<\infty$ depending only on $\gamma>1$ such that

$$
E_P\left|\sum_{j=0}^{N}\kappa^{(n)}_{jb_n+i}\right|^\gamma \le B_\gamma E_P\left|\sum_{j=0}^{N}\left(\kappa^{(n)}_{jb_n+i}\right)^2\right|^{\gamma/2} \le B_\gamma E_P\left(\sum_{j=0}^{N}\left|\kappa^{(n)}_{jb_n+i}\right|^\gamma\right) = B_\gamma(N+1)E_P|\kappa^{(n)}_1|^\gamma,
$$

where the second inequality is because $\gamma<s\le 2$ implies $\gamma/2<1$. Now, recall from [6] that $P(E_\omega T_1 > x)\sim Kx^{-s}$ for some $K>0$. Therefore, since $\gamma<s$ we have that $E_P|E_\omega T_1|^\gamma<\infty$. Thus, it's easy to see that $E_P|\kappa^{(n)}_1|^\gamma = E_P\left|E_\omega\tilde{T}^{(n)}_1 - \mathbb{E}\tilde{T}^{(n)}_1\right|^\gamma$ is uniformly bounded in n. So, there exists a constant B'_γ depending on $\gamma\in(1,s)$ such that

$$
P\left(\max_{l\le N}\left|\sum_{j=0}^{l}\kappa^{(n)}_{jb_n+i}\right| \ge \frac{n^{1-\delta'}}{3b_n}\right) \le B'_\gamma b_n^\gamma n^{-\gamma+\gamma\delta'}(N+1),
$$

and thus by (25)

$$
P\left(\max_{m\le n}\left|E_\omega\tilde{T}^{(n)}_m - m\mathbb{E}\tilde{T}^{(n)}_1\right| \ge \frac{n^{1-\delta'}}{3}\right) \le B'_\gamma b_n^{\gamma+1} n^{-\gamma+\gamma\delta'}\left(\frac{n}{b_n}+1\right) = \mathcal{O}\left(b_n^\gamma n^{1-\gamma+\gamma\delta'}\right).
$$

Since by assumption we have $\delta' < \frac{s-1}{2s}$, we may choose $\gamma<s$ arbitrarily close to s so that $b_n^\gamma n^{-\gamma+1+\gamma\delta'} = o\left(n^{-(s-1)/2}\right)$. $\square$

Proof of Proposition 1.4:
Recall the definition of $\alpha_m := n_{k_m - 1}$. To prove (11) it is enough to prove that $\forall \varepsilon > 0$

$$\lim_{m\to\infty} P_\omega\left(\left|\frac{T_{\nu_{\alpha_m}} - E_\omega T_{\nu_{\alpha_m}}}{\sqrt{v_{k_m,\omega}}}\right| \geq \varepsilon\right) = 0, \quad P-a.s. \tag{26}$$

and

$$\lim_{m\to\infty} P_\omega^{\nu_{\alpha_m}}\left(T_{x_m} \neq \bar{T}_{x_m}^{(d_{k_m})}\right) = 0, \quad \text{and} \quad \lim_{m\to\infty} E_\omega^{\nu_{\alpha_m}}\left(T_{x_m} - \bar{T}_{x_m}^{(d_{k_m})}\right) = 0, \quad P-a.s. \tag{27}$$

To prove (26), note that by Chebychev's inequality

$$P_\omega\left(\left|\frac{T_{\nu_{\alpha_m}} - E_\omega T_{\nu_{\alpha_m}}}{\sqrt{v_{k_m,\omega}}}\right| \geq \varepsilon\right) \leq \frac{Var_\omega T_{\nu_{\alpha_m}}}{\varepsilon^2 v_{k_m,\omega}},$$

which by Corollary 2.3 tends to zero $P-a.s.$ as $m \to \infty$. Secondly, to prove (27), note that since

$$P_\omega^{\nu_{\alpha_m}}\left(T_{x_m} \neq \bar{T}_{x_m}^{(d_{k_m})}\right) = P_\omega^{\nu_{\alpha_m}}\left(T_{x_m} - \bar{T}_{x_m}^{(d_{k_m})} \geq 1\right) \leq E_\omega^{\nu_{\alpha_m}}\left(T_{x_m} - \bar{T}_{x_m}^{(d_{k_m})}\right),$$

it is enough to prove only the second claim (27). However, since $x_m \leq 2n_{k_m}$ for all m large enough, it is enough to prove

$$\lim_{k\to\infty} E_\omega\left(T_{2n_k} - \bar{T}_{2n_k}^{(d_k)}\right) = 0, \quad P-a.s. \tag{28}$$

To prove (28), note that for any $\varepsilon > 0$ that

$$P\left(E_\omega\left(T_{2n_k} - \bar{T}_{2n_k}^{(d_k)}\right) \geq \varepsilon\right) \leq \frac{\mathbb{E}\left(T_{2n_k} - \bar{T}_{2n_k}^{(d_k)}\right)}{\varepsilon} \leq \frac{\mathbb{E}\left(T_{2n_k} - \tilde{T}_{2n_k}^{(d_k)}\right)}{\varepsilon} = \frac{2n_k\mathbb{E}\left(T_1 - \tilde{T}_1^{(d_k)}\right)}{\varepsilon}. \tag{29}$$

However, from (24) we have that $\mathbb{E}\left(T_1 - \tilde{T}_1^{(d_k)}\right) = \frac{2}{1-E_P\rho}(E_P\rho)^{b_{d_k}}$ which decreases faster than any power of n_k (since $E_P\rho < 1$ and $d_k \sim n_k$), and thus the last term in (29) is summable. Therefore, applying the Borel-Cantelli Lemma gives (28) which completes the proof of (11). Note, moreover, that the convergence in (11) must be uniform in y since F is continuous.

To prove (12), for any $y \in \mathbb{R}$ let $x_m(y) := \lceil n_{k_m} + y\, v_P \sqrt{v_{k_m,\omega}} \rceil$, and define $X_t^* := \max\{X_n : n \leq t\}$. Then we have

$$\begin{aligned} P_\omega\left(\frac{X_{t_m}^* - n_{k_m}}{v_P\sqrt{v_{k_m,\omega}}} < y\right) &= P_\omega\left(X_{t_m}^* < x_m(y)\right) = P_\omega\left(T_{x_m(y)} > t_m\right) \\ &= P_\omega\left(\frac{T_{x_m(y)} - E_\omega T_{x_m(y)}}{\sqrt{v_{k_m,\omega}}} > \frac{t_m - E_\omega T_{x_m(y)}}{\sqrt{v_{k_m,\omega}}}\right) \end{aligned} \tag{30}$$

Now, recalling the definition of $t_m := \lfloor E_\omega X_{n_{k_m}} \rfloor$, by Lemma 2.4 we have

$$\lim_{m\to\infty} \frac{t_m - E_\omega T_{x_m(y)}}{\sqrt{v_{k_m,\omega}}} = \lim_{m\to\infty} \frac{\lfloor E_\omega T_{n_{k_m}} \rfloor - E_\omega T_{n_{k_m} + y v_P \sqrt{v_{k_m,\omega}}}}{\sqrt{v_{k_m,\omega}}} = -y, \quad \forall y \in \mathbb{R} \quad P-a.s.,$$

where we used the fact that $v_P \mathbb{E}T_1 = 1$ due to (2). Also, by Corollary 2.2 we have $P-a.s.$ that $\sqrt{v_{k,\omega}} = o(d_k) = o(n_k)$ since $s < 2$, and therefore $x_m(y) \sim n_{k_m}$. Thus since the convergence in (11) is uniform in y, (30) gives that

$$\lim_{m\to\infty} P_\omega\left(\frac{X_{t_m}^* - n_{k_m}}{v_P\sqrt{v_{k_m,\omega}}} < y\right) = 1 - F(-y), \quad \forall y \in \mathbb{R} \quad P-a.s. \tag{31}$$

Now, (2) gives that $t_m \sim (\mathbb{E}T_1)n_{k_m}$, $P-a.s.$ Therefore, an easy argument involving [9, Lemma 4.6] and (13) gives that $X^*_{t_m} - X_{t_m} = o(\log^2 t_m) = o(\log^2 n_{k_m})$, $\mathbb{P}-a.s.$ Also, Corollary 2.2 and the Borel-Cantelli Lemma give $P-a.s.$ that $v_{k,\omega} \geq d_k^{2/s-\delta} \sim n_k^{2/s-\delta}$ for any $\delta > 0$ and all k large enough. Therefore, $\mathbb{P}-a.s.$ we have that $\lim_{m\to\infty} \frac{X^*_{t_m}-X_{t_m}}{\sqrt{v_{k_m,\omega}}} = 0$. Combining this with (31) completes the proof of (12). $\square$

Remark: For the last conclusion of Proposition 1.4 to hold it is crucial that $s > 1$. The dual nature of X_t^* and T_n always allows the transfer of probabilities from time to space. However, if $s \leq 1$ then $\mathbb{E}T_1 = \infty$ and the averaging behavior of Lemma 2.4 does not occur.

3 Quenched CLT Along a Subsequence

For the remainder of the paper we will fix the sequence $n_k := 2^{2^k}$ and let d_k and $v_{k,\omega}$ be defined accordingly as in (10). Note that this choice of n_k satisfies the conditions in Proposition 1.4 for any $\delta < 1$ since $n_k = n_{k-1}^2$. Our first goal in this section is to prove the following theorem, which when applied to Proposition 1.4 proves Theorem 1.1.

Theorem 3.1. *Assume $s < 2$. Then for any $\eta \in (0,1)$, $P-a.s.$ there exists a subsequence $n_{k_m} = n_{k_m}(\omega,\eta)$ of $n_k = 2^{2^k}$ such that for α_m, β_m and γ_m defined by*

$$\alpha_m := n_{k_m-1}, \quad \beta_m := n_{k_m-1} + \lfloor \eta d_{k_m} \rfloor, \quad \textit{and} \quad \gamma_m := n_{k_m} \tag{32}$$

and any sequence $x_m \in (\nu_{\beta_m}, \nu_{\gamma_m}]$ we have

$$\lim_{m\to\infty} P_\omega^{\nu_{\alpha_m}} \left(\frac{\bar{T}_{x_m}^{(d_{k_m})} - E_\omega \bar{T}_{x_m}^{(d_{k_m})}}{\sqrt{v_{k_m,\omega}}} \leq x \right) = \Phi(x).$$

The proof of Theorem 3 is similar to the proof of [9, Theorem 5.10]. The key is to find a random subsequence where none of the variances $\sigma^2_{i,d_{k_m},\omega}$ with $i \in (n_{k_m-1}, n_{k_m}]$ is larger than a fraction of $v_{k_m,\omega}$. To this end, let $\#(I)$ denote the cardinality of the set I, and for any $\eta \in (0,1)$ and any positive integer $a < n/2$ define the events

$$\mathcal{S}_{\eta,n,a} := \bigcup_{\substack{I \subset [1,\eta n] \\ \#(I) = 2a}} \left(\bigcap_{i\in I} \left\{ \mu^2_{i,n,\omega} \in [n^{2/s}, 2n^{2/s}) \right\} \bigcap_{j\in[1,\eta n]\backslash I} \left\{ \mu^2_{j,n,\omega} < n^{2/s} \right\} \right).$$

and

$$U_{\eta,n} := \left\{ \sum_{i\in(\eta n,n]} \sigma^2_{i,n,\omega} < 2n^{2/s} \right\}.$$

On the event $\mathcal{S}_{\eta,n,a}$, $2a$ of the first ηn crossings times from ν_{i-1} to ν_i have roughly the same size variance and the rest are all smaller. Define

$$a_k := \lfloor \log\log k \rfloor \vee 1. \tag{33}$$

Then, we have the following Lemma:

Lemma 3.2. *Assume $s < 2$. Then for any $\eta \in (0,1)$, we have $Q\left(\mathcal{S}_{\eta,d_k,a_k} \cap U_{\eta,d_k}\right) \geq \frac{1}{k}$ for all k large enough.*

Proof. First we reduce the problem to getting a lower bound on $Q(\mathcal{S}_{\eta,d_k,a_k})$. Define

$$\tilde{U}_{\eta,n} := \left\{ \sum_{i\in(\eta n+b_n,n]} \sigma^2_{i,n,\omega} < n^{2/s} \right\}.$$

Note that $\mathcal{S}_{\eta,n,a}$ and $\tilde{U}_{\eta,n}$ are independent events since $\tilde{U}_{\eta,n}$ only depends on the environment to the right of the $\nu_{\lceil \eta n\rceil}$. Thus,

$$\begin{aligned} Q\left(\mathcal{S}_{\eta,n,a} \cap U_{\eta,n}\right) &\geq Q\left(\mathcal{S}_{\eta,n,a} \cap \tilde{U}_{\eta,n}\right) - Q\left(\sum_{i\in(\eta n,\eta n+b_n]} \sigma^2_{i,n,\omega} > n^{2/s}\right) \\ &\geq Q\left(\mathcal{S}_{\eta,n,a}\right) Q\left(\tilde{U}_{\eta,n}\right) - b_n Q\left(Var_\omega \bar{T}^{(n)}_\nu > \frac{n^{2/s}}{b_n}\right). \end{aligned}$$

Now, Theorem 1.3 gives that $Q\left(\tilde{U}_{\eta,n}\right) \geq Q\left(Var_\omega T_{\nu_n} < n^{2/s}\right) = L_{\frac{s}{2},b}(1) + o(1)$, and Theorem 1.5 gives that $b_n Q\left(Var_\omega \bar{T}^{(n)}_\nu > \frac{n^{2/s}}{b_n}\right) \sim K_\infty b_n^{1+s} n^{-1}$. Thus,

$$Q\left(\mathcal{S}_{\eta,d_k,a_k} \cap U_{\eta,d_k}\right) \geq Q(\mathcal{S}_{\eta,d_k,a_k})(L_{\frac{s}{2},b}(1) + o(1)) - \mathcal{O}(b_{d_k}^{1+s} d_k^{-1}), \quad \text{as } k \to \infty,$$

and so to prove the lemma it is enough to show that $\lim_{k\to\infty} k\, Q(\mathcal{S}_{\eta,d_k,a_k}) = \infty$. A lower bound for $Q(\mathcal{S}_{\eta,n,a})$ was derived in [9, preceeding Lemma 5.7]. A similar argument gives that for any $\varepsilon < \frac{1}{3}$ there exists a constant $C_\varepsilon > 0$ such that

$$\begin{aligned} Q\left(\mathcal{S}_{\eta,n,a}\right) \geq &\frac{(\eta C_\varepsilon)^{2a}}{(2a)!} \left(1 - \frac{(2a-1)(1+4b_n)}{\eta n}\right)^{2a} \left(Q\left(\sum_{i=1}^n \left(E_\omega^{\nu_{i-1}} T_{\nu_i}\right)^2 < n^{2/s}\right) - a\, o(n^{-1+2\varepsilon})\right) \\ &- \frac{(\eta n)^{2a}}{(2a)!} a\, o\left(e^{-n^{\varepsilon/(6s)}}\right), \end{aligned} \tag{34}$$

where asymptotics of the form $o(\cdot\,)$ in (34) are uniform in η and a as $n\to\infty$. The proof of (34) is exactly the same as in [9] with the exception that the lower bound for $Q\left(\bigcap_{j\in[1,n]} \left\{\mu^2_{j,n,\omega} < n^{2/s}\right\}\right)$ in [9, (68)] is $Q\left(\sum_{i=1}^n \left(E_\omega^{\nu_{i-1}} T_{\nu_i}\right)^2 < n^{2/s}\right)$ instead of $Q\left(E_\omega T_{\nu_n} < n^{1/s}\right)$. Then, replacing n and a in (34) by d_k and a_k respectively, we have for $\varepsilon < \frac{1}{3}$ that

$$\begin{aligned} Q\left(\mathcal{S}_{\eta,d_k,a_k}\right) \geq &\frac{(\eta C_\varepsilon)^{2a_k}}{(2a_k)!} \left(1 - \frac{(2a_k-1)(1+4b_{d_k})}{\eta d_k}\right)^{2a_k} \left(Q\left(\sum_{i=1}^{d_k} \left(E_\omega^{\nu_{i-1}} T_{\nu_i}\right)^2 < d_k^{2/s}\right) - a_k o(d_k^{-1+2\varepsilon})\right) \\ &- \frac{(\eta d_k)^{2a_k}}{(2a_k)!} a_k o\left(e^{-d_k^{\varepsilon/(6s)}}\right) \\ = &\frac{(\eta C_\varepsilon)^{2a_k}}{(2a_k)!} (1+o(1)) \left(L_{\frac{s}{2},b}(1) - o(1)\right) - o\left(\frac{1}{k}\right). \end{aligned} \tag{35}$$

The last equality is a result of Theorem 1.3 and the definitions of a_k and d_k in (33) and (10). Also, since $a_k \sim \log\log k$ we have that $\lim_{k\to\infty} k \frac{C^{2a_k}}{(2a_k)!} = \infty$ for any constant $C > 0$. Therefore, (35) implies that $\lim_{k\to\infty} k\, Q\left(\mathcal{S}_{\eta,d_k,a_k}\right) = \infty$. $\square$

Corollary 3.3. *Assume $s<2$. Then for any $\eta \in (0,1)$, P-a.s. there exists a random subsequence $n_{k_m} = n_{k_m}(\omega,\eta)$ of $n_k = 2^{2^k}$ such that for the sequences α_m, β_m, and γ_m defined as in (32) we have that for all m*

$$\max_{i\in(\alpha_m,\beta_m]} \mu^2_{i,d_{k_m},\omega} \leq 2d_{k_m}^{2/s} \leq \frac{1}{a_{k_m}} \sum_{i=\alpha_m+1}^{\beta_m} \mu^2_{i,d_{k_m},\omega}, \quad \text{and} \quad \sum_{i=\beta_m+1}^{\gamma_m} \sigma^2_{i,d_{k_m},\omega} < 2d_{k_m}^{2/s}. \tag{36}$$

Proof. Define the sequence of events

$$\mathcal{S}'_k := \bigcup_{\substack{I \subset (n_{k-1},\, n_{k-1}+\eta d_k] \\ \#(I) = 2a_k}} \left(\bigcap_{i\in I} \left\{ \mu^2_{i,d_k,\omega} \in [d_k^{2/s}, 2d_k^{2/s}) \right\} \bigcap_{j\in(n_{k-1},n_{k-1}+\eta d_k]\backslash I} \left\{ \mu^2_{j,d_k,\omega} < d_k^{2/s} \right\} \right),$$

and

$$U'_k := \left\{ \sum_{i\in(n_{k-1}+\eta d_k, n_k]} \sigma^2_{i,d_{k_m},\omega} < 2d_{k_m}^{2/s} \right\}$$

Note that due to the reflections of the random walk, the event $\mathcal{S}'_k \cap U'_k$ depends on the environment between ladder locations $n_{k-1} - b_{d_k}$ and n_k. Thus, since $n_{k-1} - b_{d_k} > n_{k-2}$ for all $k \geq 4$, we have that $\{\mathcal{S}'_{2k} \cap U'_{2k}\}_{k=2}^\infty$ is an independent sequence of events. Similarly, for k large enough $\mathcal{S}'_k \cap U'_k$ does not depend on the environment to left of the origin. Thus

$$P(\mathcal{S}'_k \cap U'_k) = Q(\mathcal{S}'_k \cap U'_k) = Q\left(\mathcal{S}_{\eta,d_k,a_k} \cap U_{\eta,d_k}\right)$$

for all k large enough. Lemma 3.2 then gives that $\sum_{k=1}^\infty P(\mathcal{S}'_{2k} \cap U'_{2k}) = \infty$, and the Borel-Cantelli Lemma then implies that infinitely many of the events $\mathcal{S}'_{2k} \cap U'_{2k}$ occur $P-a.s.$ Therefore, $P-a.s.$ there exists a subsequence $k_m = k_m(\omega,\eta)$ such that $\mathcal{S}'_{k_m} \cap U'_{k_m}$ occurs for each m. Finally, note that the event $\mathcal{S}'_{k_m} \cap U'_{k_m}$ implies (36). □

Proof of Theorem 3.1:
First, recall that [9, Corollary 5.6] gives that there exists an $\eta' > 0$ such that

$$Q\left(\left| \sum_{i=1}^n \left(\sigma^2_{i,m,\omega} - \mu^2_{i,m,\omega}\right) \right| \geq \delta n^{2/s} \right) = o(n^{-\eta'}) \qquad \forall \delta > 0, \quad \forall m \in \mathbb{N}. \tag{37}$$

This can be applied along with the Borel-Cantelli Lemma to prove that

$$\sum_{i=n_{k-1}+1}^{n_{k-1}+\lfloor \eta d_k \rfloor} \left(\sigma^2_{i,d_k,\omega} - \mu^2_{i,d_k,\omega}\right) = o\left(d_k^{2/s}\right), \quad P-a.s. \tag{38}$$

Thus, $P-a.s.$ we may assume that (38) holds and that there exists a subsequence $n_{k_m} = n_{k_m}(\omega,\eta)$ such that condition (36) in Corollary 3.3 holds. Then, it is enough to prove that

$$\lim_{m\to\infty} P_\omega^{\nu_{\alpha_m}} \left(\frac{\bar{T}^{(d_{k_m})}_{\nu_{\beta_m}} - E_\omega^{\nu_{\alpha_m}} \bar{T}^{(d_{k_m})}_{\nu_{\beta_m}}}{\sqrt{v_{k_m,\omega}}} \leq y \right) = \Phi(y), \tag{39}$$

and

$$\lim_{m\to\infty} P_\omega^{\nu_{\beta_m}}\left(\left|\frac{\bar{T}_{x_m}^{(d_{k_m})} - E_\omega^{\nu_{\beta_m}}\bar{T}_{x_m}^{(d_{k_m})}}{\sqrt{v_{k_m,\omega}}}\right| \geq \varepsilon\right) = 0, \quad \forall \varepsilon > 0. \tag{40}$$

To prove (40), note that by Chebychev's inequality

$$P_\omega^{\nu_{\beta_m}}\left(\left|\frac{\bar{T}_{x_m}^{(d_{k_m})} - E_\omega^{\nu_{\beta_m}}\bar{T}_{x_m}^{(d_{k_m})}}{\sqrt{v_{k_m,\omega}}}\right| \geq \varepsilon\right) \leq \frac{Var_\omega\left(\bar{T}_{x_m}^{(d_{k_m})} - \bar{T}_{\beta_m}^{(d_{k_m})}\right)}{\varepsilon^2 v_{k_m,\omega}} \leq \frac{\sum_{i=\beta_m+1}^{\gamma_m}\sigma^2_{i,d_{k_m},\omega}}{\varepsilon^2 v_{k_m,\omega}}$$

However, by (38) and our choice of the subsequence n_{k_m} we have that $\sum_{i=\beta_m+1}^{\gamma_m}\sigma^2_{i,d_{k_m},\omega} < 2d_{k_m}^{2/s}$, and $v_{k_m,\omega} \geq \sum_{i=\alpha_m+1}^{\beta_m}\sigma^2_{i,d_{k_m},\omega} = \sum_{i=\alpha_m+1}^{\beta_m}\mu^2_{i,d_{k_m},\omega} + o\left(d_{k_m}^{2/s}\right) \geq a_{k_m}d_{k_m}^{2/s} + o\left(d_{k_m}^{2/s}\right)$. Thus

$$\lim_{m\to\infty}\frac{\sum_{i=\beta_m+1}^{\gamma_m}\sigma^2_{i,d_{k_m},\omega}}{v_{k_m,\omega}} = 0, \tag{41}$$

which proves (40). To prove (39), it is enough to show that the Lindberg-Feller condition is satisfied. That is we need to show

$$\lim_{m\to\infty}\frac{1}{v_{k_m,\omega}}\sum_{i=\alpha_m+1}^{\beta_m}\sigma^2_{i,d_{k_m},\omega} = 1, \tag{42}$$

and

$$\lim_{m\to\infty}\frac{1}{v_{k_m,\omega}}\sum_{i=\alpha_m+1}^{\beta_m}E_\omega^{\nu_{i-1}}\left[\left(\bar{T}_{\nu_i}^{(d_{k_m})} - \mu_{i,d_{k_m},\omega}\right)^2\mathbf{1}_{|\bar{T}_{\nu_i}^{(d_{k_m})}-\mu_{i,d_{k_m},\omega}|>\varepsilon\sqrt{v_{m,\omega}}}\right] = 0, \quad \forall\varepsilon > 0. \tag{43}$$

To show (42) note that the definition of $v_{k_m,\omega}$ and our choice of the subsequence n_{k_m} give that

$$\frac{1}{v_{k_m,\omega}}\sum_{i=\alpha_m+1}^{\beta_m}\sigma^2_{i,d_{k_m},\omega} = 1 - \frac{1}{v_{k_m,\omega}}\sum_{i=\beta_m+1}^{\gamma_m}\sigma^2_{i,d_{k_m},\omega} = 1 - o(1),$$

where the last equality is from (41). To prove (43), first note that an application of [9, Lemma 5.5] gives that for any $\varepsilon' > 0$

$$\sum_{i=n_{k-1}+1}^{n_{k-1}+\lfloor \eta d_k\rfloor}\sigma^2_{i,d_k,\omega}\mathbf{1}_{M_i\leq d_k^{(1-\varepsilon')/s}} = o\left(d_k^{2/s}\right), \quad P-a.s.,$$

where M_i is defined as in (16). Then, since $v_{k_m,\omega} \geq a_{k_m}d_{k_m}^{2/s} + o\left(d_{k_m}^{2/s}\right)$ we can reduce the sum in (43) to blocks where $M_i > d_{k_m}^{(1-\varepsilon')/s}$. That is, it is enough to prove that for some $\varepsilon' > 0$ and every $\varepsilon > 0$

$$\lim_{m\to\infty}\frac{1}{v_{k_m,\omega}}\sum_{i=\alpha_m+1}^{\beta_m}E_\omega^{\nu_{i-1}}\left[\left(\bar{T}_{\nu_i}^{(d_{k_m})} - \mu_{i,d_{k_m},\omega}\right)^2\mathbf{1}_{|\bar{T}_{\nu_i}^{(d_{k_m})}-\mu_{i,d_{k_m},\omega}|>\varepsilon\sqrt{v_{k_m,\omega}}}\right]\mathbf{1}_{M_i>d_{k_m}^{(1-\varepsilon')/s}} = 0. \tag{44}$$

To get an upper bound for (44), first note that our choice of the subsequence n_{k_m} gives that for m large enough $v_{k_m,\omega} \geq \frac{1}{2}\sum_{i=\alpha_m+1}^{\beta_m}\mu^2_{i,d_{k_m},\omega} \geq \frac{a_{k_m}}{2}\mu_{i,d_{k_m},\omega}$ for any $i \in (\alpha_m, \beta_m]$. Thus, for m large enough we can replace the indicators inside the expectations in (44) by the indicators of the events $\left\{\bar{T}_{\nu_i}^{(d_{k_m})} > (1+\varepsilon\sqrt{a_{k_m}/2})\mu_{i,d_{k_m},\omega}\right\}$. Thus, for m large enough and $i \in (\alpha_m, \beta_m]$, we have

$$
\begin{aligned}
&E_\omega^{\nu_{i-1}}\left[\left(\bar{T}_{\nu_i}^{(d_{k_m})} - \mu_{i,d_{k_m},\omega}\right)^2 \mathbf{1}_{|\bar{T}_{\nu_i}^{(d_{k_m})} - \mu_{i,d_{k_m},\omega}| > \varepsilon\sqrt{v_{k_m,\omega}}}\right] \\
&\quad \le E_\omega^{\nu_{i-1}}\left[\left(\bar{T}_{\nu_i}^{(d_{k_m})} - \mu_{i,d_{k_m},\omega}\right)^2 \mathbf{1}_{\bar{T}_{\nu_i}^{(d_{k_m})} > (1+\varepsilon\sqrt{a_{k_m}/2})\mu_{i,d_{k_m},\omega}}\right] \\
&\quad = \int_{1+\varepsilon\sqrt{a_{k_m}/2}}^{\infty} P_\omega^{\nu_{i-1}}\left(\bar{T}_{\nu_i}^{(d_{k_m})} > x\mu_{i,d_{k_m},\omega}\right) 2(x-1)\mu_{i,d_{k_m},\omega}^2 \, dx\,.
\end{aligned}
$$

We want to get an upper bound on the probabilities inside the integral. If $\varepsilon' < \frac{1}{3}$ we can use [9, Lemma 5.9] to get that for k large enough, $E_\omega^{\nu_{i-1}}\left(\bar{T}_{\nu_i}^{(d_k)}\right)^j \le 2^j j! \mu_{i,d_k,\omega}^j$ for all $n_{k-1} < i \le n_k$ such that $M_i > d_k^{(1-\varepsilon')/s}$. Multiplying by $(4\mu_{i,d_k,\omega})^{-j}$ and summing over j gives that $E_\omega^{\nu_{i-1}} e^{\bar{T}_{\nu_i}^{(d_k)}/(4\mu_{i,d_k,\omega})} \le 2$. Therefore, Chebychev's inequality gives

$$
P_\omega^{\nu_{i-1}}\left(\bar{T}_{\nu_i}^{(d_k)} > x\mu_{i,d_k,\omega}\right) \le e^{-x/4} E_\omega^{\nu_{i-1}} e^{\bar{T}_{\nu_i}^{(d_k)}/(4\mu_{i,d_k,\omega})} \le 2e^{-x/4}\,.
$$

Thus, for all m large enough we have for all $\alpha_m < i \le \beta_m \le n_{k_m}$ with $M_i > d_{k_m}^{(1-\varepsilon')/s}$ that

$$
\begin{aligned}
\int_{1+\varepsilon\sqrt{a_{k_m}/2}}^{\infty} P_\omega^{\nu_{i-1}}\left(\bar{T}_{\nu_i}^{(d_{k_m})} > x\mu_{i,d_{k_m},\omega}\right) 2(x-1)\mu_{i,d_{k_m},\omega}^2 dx &\le \mu_{i,d_{k_m},\omega}^2 \int_{1+\varepsilon\sqrt{a_{k_m}/2}}^{\infty} 4(x-1)e^{-x/4} dx \\
&= \mu_{i,d_{k_m},\omega}^2\, o\left(e^{-a_{k_m}^{1/4}}\right)\,.
\end{aligned}
$$

Therefore we have that as $m \to \infty$, (44) is bounded above by

$$
\lim_{m\to\infty} o\left(e^{-a_{k_m}^{1/4}}\right) \frac{1}{v_{k_m,\omega}} \left(\sum_{i=\alpha_m+1}^{\beta_m} \mu_{i,d_{k_m},\omega}^2 \mathbf{1}_{M_i > d_{k_m}^{(1-\varepsilon')/s}}\right)\,. \tag{45}
$$

However, since

$$
\frac{1}{v_{k_m,\omega}} \sum_{i=\alpha_m+1}^{\beta_m} \mu_{i,d_{k_m},\omega}^2 \le \frac{1}{\sum_{i=\alpha_m+1}^{\beta_m} \sigma_{i,d_{k_m},\omega}^2} \left(\sum_{i=\alpha_m+1}^{\beta_m} \sigma_{i,d_{k_m},\omega}^2 + o\left(d_{k_m}^{2/s}\right)\right) \le 1 + \frac{o\left(d_{k_m}^{2/s}\right)}{2a_{k_m} d_{k_m}^{2/s} + o\left(d_{k_m}^{2/s}\right)},
$$

we have that (45) tends to zero as $m \to \infty$. This finishes the proof of (43) and thus of Theorem 3.1. □

Proof of Theorem 1.1:
Choose $\eta \in (0,1)$ such that $\eta < \frac{1}{\bar{\nu}}$ where $\bar{\nu} = E_P \nu$, and then choose n_{k_m} as in Theorem 3.1. Then for β_m and γ_m defined as in (32), we have that (14) and the fact that $d_k \sim n_k$ give

$$
\lim_{m\to\infty} \frac{\nu_{\beta_m}}{n_{k_m}} = \eta\bar{\nu} < 1 < \bar{\nu} = \lim_{m\to\infty} \frac{\nu_{\gamma_m}}{n_{k_m}}.
$$

Thus $x_m \sim n_{k_m} \Rightarrow x_m \in [\nu_{\beta_m}, \nu_{\gamma_m}]$ for all m large enough. Therefore, the conditions of Proposition 1.4 are satisfied with $F(x) = \Phi(x)$. □

4 Quenched Exponential Limits

4.1 Analysis of T_ν when M_1 is Large

The goal of this subsection is to analyze the quenched distribution of $\bar{T}_\nu^{(n)}$ on "large" blocks (i.e. when $M_1 > n^{(1-\varepsilon)/s}$). We want to show that conditioned on M_1 being large, $\bar{T}_\nu^{(n)}/E_\omega \bar{T}_\nu^{(n)}$ is approximately exponentially distributed. We do this by showing that the quenched Laplace transform $E_\omega \exp\left\{-\lambda \frac{\bar{T}_\nu^{(n)}}{E_\omega \bar{T}_\nu^{(n)}}\right\}$ is approximately $\frac{1}{1+\lambda}$ on such blocks.

As was done in [2], we analyze the quenched Laplace transform of $\bar{T}_\nu^{(n)}$ by decomposing $\bar{T}_\nu^{(n)}$ into a series of excursions away from 0. An excursion is a "failure" if the random walk returns to zero before hitting ν (i.e. if $T_\nu > T_0^+ := \min\{k > 0 : X_k = 0\}$), and a "success" if the random walk reaches ν before returning to zero (note that classifying an excursion as a failure/sucess is independent of any modifications to the environment left of zero since if the random walk ventures to the left at all, it must be in a failure excursion). Define $p_\omega := P_\omega(T_\nu < T_0^+)$, and let N be a geometric random variable with parameter p_ω (i.e. $P(N = k) = p_\omega(1 - p_\omega)^k$ for $k \in \mathbb{N}$). Also, let $\{F_i\}_{i=1}^\infty$ be an i.i.d. sequence (also independent of N) with F_1 having the same distribution as $\bar{T}_\nu^{(n)}$ conditioned on $\left\{\bar{T}_\nu^{(n)} > T_0^+\right\}$, and let S be a random variable with the same distribution as T_ν conditioned on $\left\{T_\nu < T_0^+\right\}$ and independent of everything else (note that for sucess excursions we can ignore added reflections to the left of zero). Thus, we have that

$$\bar{T}_\nu^{(n)} \stackrel{Law}{=} S + \sum_{i=1}^{N} F_i \qquad \text{(quenched)}. \tag{46}$$

In a slight abuse of notation we will still use P_ω for the probabilities of F_i, S, and N to emphasize that their distributions are dependent on ω. The following results are easy to verify:

$$E_\omega N = \frac{1-p_\omega}{p_\omega} \quad \text{and} \quad E_\omega \bar{T}_\nu^{(n)} = E_\omega S + (E_\omega N)(E_\omega F_1), \tag{47}$$

$$\begin{aligned} Var_\omega \bar{T}_\nu^{(n)} &= (E_\omega N)(Var_\omega F_1) + (E_\omega F)^2 (Var_\omega N) + Var_\omega S \\ &= (E_\omega N)(E_\omega F^2) + (E_\omega F)^2 (Var_\omega N - E_\omega N) + Var_\omega S \\ &= (E_\omega N)(E_\omega F^2) + (E_\omega F)^2 (E_\omega N)^2 + Var_\omega S, \end{aligned} \tag{48}$$

and

$$E_\omega e^{-\lambda \bar{T}_\nu^{(n)}} = E_\omega e^{-\lambda S} E_\omega\left[\left(E_\omega e^{-\lambda F_1}\right)^N\right] = E_\omega e^{-\lambda S} \frac{p_\omega}{1-(1-p_\omega)\left(E_\omega e^{-\lambda F_1}\right)}, \quad \forall \lambda \geq 0.$$

Also, since $e^{-x} \geq 1 - x$ for any $x \in \mathbb{R}$ we have for any $\lambda \geq 0$ that

$$E_\omega e^{-\lambda \bar{T}_\nu^{(n)}} \geq (1 - \lambda E_\omega S) \frac{p_\omega}{1-(1-p_\omega)(1-\lambda E_\omega F_1)} = \frac{1-\lambda E_\omega S}{1+\lambda (E_\omega N)(E_\omega F_1)} \geq \frac{1-\lambda E_\omega S}{1+\lambda E_\omega \bar{T}_\nu^{(n)}},$$

where the first equality and the last inequality are from the formulas for $E_\omega N$ and $E_\omega \bar{T}_\nu^{(n)}$ given in (47). Similarly, since $e^{-x} \le 1 - x + \frac{x^2}{2}$ for all $x \ge 0$ we have that for any $\lambda \ge 0$ that

$$\begin{aligned} E_\omega e^{-\lambda \bar{T}_\nu^{(n)}} &\le \frac{p_\omega}{1-(1-p_\omega)\left(1-\lambda E_\omega F_1 + \frac{\lambda^2}{2} E_\omega F_1^2\right)} \\ &= \frac{1}{1+\lambda(E_\omega N)(E_\omega F_1) - \frac{\lambda^2}{2}(E_\omega N)(E_\omega F_1^2)} \\ &= \frac{1}{1+\lambda(E_\omega N)(E_\omega F_1) - \frac{\lambda^2}{2}(Var_\omega \bar{T}_\nu^{(n)} - (E_\omega N)^2(E_\omega F_1)^2 - Var_\omega S)} \\ &\le \frac{1}{1+\lambda(E_\omega \bar{T}_\nu^{(n)} - E_\omega S) - \frac{\lambda^2}{2}(Var_\omega \bar{T}_\nu^{(n)} - (E_\omega \bar{T}_\nu^{(n)} - E_\omega S)^2)}, \end{aligned}$$

where the first equality and last inequality are from (47) and the second equality is from (48). Therefore, replacing λ by $\lambda/(E_\omega \bar{T}_\nu^{(n)})$ we get

$$E_\omega e^{-\lambda \frac{\bar{T}_\nu^{(n)}}{E_\omega \bar{T}_\nu^{(n)}}} \ge \left(1 - \lambda \frac{E_\omega S}{E_\omega \bar{T}_\nu^{(n)}}\right) \frac{1}{1+\lambda}\,, \tag{49}$$

and

$$\begin{aligned} E_\omega e^{-\lambda \frac{\bar{T}_\nu^{(n)}}{E_\omega \bar{T}_\nu^{(n)}}} &\le \frac{1}{1+\lambda-\lambda\frac{E_\omega S}{E_\omega \bar{T}_\nu^{(n)}} - \frac{\lambda^2}{2}\left(\frac{Var_\omega \bar{T}_\nu^{(n)}}{(E_\omega \bar{T}_\nu^{(n)})^2} - \frac{(E_\omega \bar{T}_\nu^{(n)} - E_\omega S)^2}{(E_\omega \bar{T}_\nu^{(n)})^2}\right)} \\ &\le \frac{1}{1+\lambda-(\lambda+\lambda^2)\frac{E_\omega S}{E_\omega \bar{T}_\nu^{(n)}} - \frac{\lambda^2}{2}\left(\frac{Var_\omega \bar{T}_\nu^{(n)}}{(E_\omega \bar{T}_\nu^{(n)})^2} - 1\right)}\,. \end{aligned} \tag{50}$$

Therefore, we have reduced the problem of showing $E_\omega e^{-\lambda \frac{\bar{T}_\nu^{(n)}}{E_\omega \bar{T}_\nu^{(n)}}} \approx \frac{1}{1+\lambda}$ when M_1 is large to showing that $\frac{E_\omega S}{E_\omega \bar{T}_\nu^{(n)}} \approx 0$ and $\frac{Var_\omega \bar{T}_\nu^{(n)}}{(E_\omega \bar{T}_\nu^{(n)})^2} \approx 1$ when M_1 is large. In order to analyze $E_\omega S$, we define a modified environment which is essentially the environment the random walker "sees" once it is told that it reaches ν before returning to zero. A simple computation similar to the one in [12, Remark 2 on pages 222-223] gives that the random walk conditioned to reach ν before returning to zero is a homogeneous markov chain with transition probabilities given by $\bar{\omega}_i := P_\omega^i(X_1 = i+1 | T_\nu < T_0^+)$. Then the definition of $\bar{\omega}_i$ gives that $\bar{\omega}_0 = \bar{\omega}_1 = 1$, and for $i \in [2,\nu)$ we have $\bar{\omega}_i = \frac{\omega_i P_\omega^{i+1}(T_\nu < T_0)}{P_\omega^i(T_\nu < T_0)}$. Using the hitting time formulas in [12, (2.1.4)] we have

$$\bar{\omega}_i = \frac{\omega_i R_{0,i}}{R_{0,i-1}} \quad \forall i \in [2,\nu), \quad \text{where} \quad R_{0,i} := \sum_{j=0}^{i} \Pi_{0,j}. \tag{51}$$

Let $\bar{\rho}_i := \frac{1-\bar{\omega}_i}{\bar{\omega}_i}$ and define $\bar{\Pi}_{i,j}$, and $\bar{W}_{i,j}$ analogously to $\Pi_{i,j}$ and $W_{i,j}$ using $\bar{\rho}_i$ in place of ρ_i. Then the above formulas for $\bar{\omega}_i$ give that $\bar{\rho}_0 = \bar{\rho}_1 = 0$ and $\bar{\rho}_i = \rho_i \frac{R_{0,i-2}}{R_{0,i}} \quad \forall i \in [2,\nu)$. Thus,

$$\bar{\Pi}_{i,j} = \Pi_{i,j} \frac{R_{0,i-2} R_{0,i-1}}{R_{0,j-1} R_{0,j}}, \quad \forall 2 \le i \le j < \nu. \tag{52}$$

Note that since $R_{0,i} \le R_{0,j}$ for any $0 \le i \le j$ we have from (52) that

$$\bar{\Pi}_{i,j} \le \Pi_{i,j} \quad \text{for any } 0 \le i \le j < \nu \tag{53}$$

Now, since $E_\omega S = E_{\bar{\omega}} T_\nu$ we get from (15) that $E_\omega S = \nu + 2\sum_{j=2}^{\nu-1} \bar{W}_{2,j} = \nu + 2\sum_{j=2}^{\nu-1}\sum_{i=2}^{j} \bar{\Pi}_{i,j}$. Therefore, letting $\bar{M}_1 := \max\{\bar{\Pi}_{i,j} : 0 \le i \le j < \nu\}$ we get the bound

$$E_\omega S \le \nu + 2\nu^2 \bar{M}_1. \tag{54}$$

Thus, we need to get bounds on the tail of $\bar{M}_1$. To this end, recall the definition of M_1 in (16) and define $\tau := \max\{k \in [1,\nu] : \Pi_{0,k-1} = M_1\}$. Then, define

$$M^- := \min\{\Pi_{i,j} : 0 < i \le j < \tau\} \wedge 1, \quad \text{and} \quad M^+ := \max\{\Pi_{i,j} : \tau < i \le j < \nu\} \vee 1\,. \tag{55}$$

Lemma 4.1. *For any $\varepsilon, \delta > 0$ we have*

$$P(M^- < n^{-\delta}, M_1 > n^{(1-\varepsilon)/s}) = o(n^{-1+\varepsilon-\delta s+\varepsilon'}), \quad \forall \varepsilon' > 0, \tag{56}$$

and

$$P(M^+ > n^{\delta}, M_1 > n^{(1-\varepsilon)/s}) = o(n^{-1+\varepsilon-\delta s+\varepsilon'}), \quad \forall \varepsilon' > 0, \tag{57}$$

Proof. Since $\Pi_{0,\tau-1} = M_1$ by definition we have

$$\begin{aligned} P(M^- < n^{-\delta}, M_1 > n^{(1-\varepsilon)/s}) &\le P\left(\exists 0 < i \le j < \tau - 1 : \Pi_{i,j} < n^{-\delta}, \quad \Pi_{0,\tau-1} > n^{(1-\varepsilon)/s}\right) \\ &\le P(\tau > b_n) + \sum_{0<i\le j<k<b_n} P\left(\Pi_{i,j} < n^{-\delta}, \quad \Pi_{0,k} > n^{(1-\varepsilon)/s}\right) \\ &\le P(\nu > b_n) + \sum_{0<i\le j<k<b_n} P\left(\Pi_{0,i-1}\Pi_{j+1,k} > n^{(1-\varepsilon)/s+\delta}\right). \end{aligned} \tag{58}$$

Since (13) gives that $P(\nu > b_n) \le C_1 e^{-C_2 b_n}$ we need only handle the second term in (58) to prove (56). However, Chebychev's inequality and the fact that P is a product measure give that

$$P\left(\Pi_{0,i-1}\Pi_{j+1,k} > n^{(1-\varepsilon)/s+\delta}\right) \le n^{-1+\varepsilon-\delta s}(E_P \rho^s)^{i+k-j} = n^{-1+\varepsilon-\delta s}.$$

Since the number of terms in the sum in (58) is at most $(b_n)^3 = o(n^{\varepsilon'})$ we have proved (56). The proof of (57) is similar:

$$\begin{aligned} P(M^+ > n^{\delta}, M_1 > n^{(1-\varepsilon)/s}) &\le P\left(\exists \tau < i \le j < \nu : \Pi_{i,j} > n^{\delta}, \quad \Pi_{0,\tau-1} > n^{(1-\varepsilon)/s}\right) \\ &\le P(\nu > b_n) + \sum_{0\le k<i\le j<b_n} P\left(\Pi_{0,k}\Pi_{i,j} > n^{(1-\varepsilon)/s+\delta}\right) \\ &\le C_1 e^{-C_2 b_n} + (b_n)^3 n^{-1+\varepsilon-\delta s} = o(n^{-1+\varepsilon-\delta s+\varepsilon'}) \end{aligned}$$

$\square$

Corollary 4.2. *For any $\varepsilon, \delta > 0$ we have*

$$P\left(E_\omega S \ge n^{5\delta}, M_1 > n^{(1-\varepsilon)/s}\right) = o(n^{-1+\varepsilon-\delta s+\varepsilon'}), \quad \forall \varepsilon' > 0.$$

Proof. Recall that (54) gives $E_\omega S \le \nu + 2\nu^2 \bar{M}_1$. We will use M^- and M^+ to get bounds on $\bar{M}_1$. First, note that for any $i \in [0, \tau)$ we have

$$R_{0,i} = \sum_{k=0}^{i} \Pi_{0,k} = \Pi_{0,i} + \sum_{k=0}^{i-1} \frac{\Pi_{0,i}}{\Pi_{k+1,i}} \le \Pi_{0,i} \left(\frac{i+1}{M^-} \right).$$

Note also that $R_{0,j} \ge \Pi_{0,j}$ holds for any $j \ge 0$. Thus, for any $2 \le i \le j \le \tau$ we have

$$\bar{\Pi}_{i,j} = \Pi_{i,j} \frac{R_{0,i-2} R_{0,i-1}}{R_{0,j-1} R_{0,j}} \le \Pi_{i,j} \left(\frac{i}{M^-} \right)^2 \frac{\Pi_{0,i-2}\Pi_{0,i-1}}{\Pi_{0,j-1}\Pi_{0,j}} = \left(\frac{i}{M^-} \right)^2 \frac{1}{\Pi_{i-1,j-1}} \le \frac{i^2}{(M^-)^3}.$$

Also, from (53) we have that $\bar{\Pi}_{i,j} \le \Pi_{i,j} \le M^+$ for $\tau < i \le j < \nu$. Therefore we have that $\bar{M}_1 \le \frac{\nu^2 M^+}{(M^-)^3}$ (note that here we used that $M^- \le 1$ and $M^+ \ge 1$). Thus,

$$P\left(E_\omega S \ge n^{5\delta}, M_1 > n^{(1-\varepsilon)/s} \right) \le P\left(\nu + \frac{2\nu^4 M^+}{(M^-)^3} > n^{5\delta}, M_1 > n^{(1-\varepsilon)/s} \right).$$

An easy argument using (13) and Lemma 4.1 finishes the proof. □

Lemma 4.3. *For any $\varepsilon, \delta > 0$ we have*

$$Q\left(\left| \frac{Var_\omega \bar{T}_\nu^{(n)}}{(E_\omega \bar{T}_\nu^{(n)})^2} - 1 \right| \ge n^{-\delta}, \quad M_1 > n^{(1-\varepsilon)/s} \right) = o(n^{-2+2\varepsilon+\delta s+\varepsilon'}), \quad \forall \varepsilon' > 0$$

Proof. Recall that from [9, (59)] we have that there exist explicit non-negative random variables $D^+(\omega)$ and $D^-(\omega)$ such that

$$\left(E_\omega \bar{T}_\nu^{(n)} \right)^2 - D^+(\omega) \le Var_\omega \bar{T}_\nu^{(n)} \le \left(E_\omega \bar{T}_\nu^{(n)} \right)^2 + 8 R_{0,\nu-1} D^-(\omega),$$

where $R_{0,\nu-1}$ is defined as in (51). Therefore, since $E_\omega \bar{T}_\nu^{(n)} \ge M_1$, we have

$$\begin{aligned} Q&\left(\left| \frac{Var_\omega \bar{T}_\nu^{(n)}}{(E_\omega \bar{T}_\nu^{(n)})^2} - 1 \right| \ge n^{-\delta}, M_1 > n^{(1-\varepsilon)/s} \right) \\ &\le Q\left(8 R_{0,\nu-1} D^-(\omega) > n^{(2-2\varepsilon)/s-\delta} \right) + Q\left(D^+(\omega) > n^{(2-2\varepsilon)/s-\delta} \right). \end{aligned} \tag{59}$$

However, [9, Lemma 5.2 & Corollary 5.4] give respectively that $Q(D^+(\omega) > x) = o(x^{-s+\varepsilon''})$ and $Q\left(R_{0,\nu-1} D^-(\omega) > x\right) = o(x^{-s+\varepsilon''})$ for any $\varepsilon'' > 0$. Therefore, both terms in (59) are of order $o\left(n^{-2+2\varepsilon+\delta s+\varepsilon''((2-2\varepsilon)/s-\delta)} \right)$. The lemma then follows since $\varepsilon'' > 0$ is arbitrary. □

For any i, define the scaled quenched Laplace transforms $\phi_{i,n}(\lambda) := E_\omega^{\nu_{i-1}} \exp\left\{ -\lambda \frac{\bar{T}_{\nu_i}^{(n)}}{\mu_{i,n,\omega}} \right\}$.

Lemma 4.4. *Let $\varepsilon < \frac{1}{8}$, and define $\varepsilon' := \frac{1-8\varepsilon}{5} > 0$. Then*

$$Q\left(\exists \lambda \ge 0 : \phi_{1,n}(\lambda) \notin \left[\frac{1 - \lambda n^{-\varepsilon/s}}{1+\lambda}, \frac{1}{1 + \lambda - \left(\lambda + \frac{3\lambda^2}{2} \right) n^{-\varepsilon/s}} \right], \; M_1 > n^{(1-\varepsilon)/s} \right) = o\left(n^{-1-\varepsilon'} \right).$$

Proof. Recall from (49) and (49) that

$$\left(1-\lambda\frac{E_\omega S}{E_\omega \bar{T}_\nu^{(n)}}\right)\frac{1}{1+\lambda} \leq \phi_{1,n}(\lambda) \leq \frac{1}{1+\lambda-(\lambda+\lambda^2)\frac{E_\omega S}{E_\omega \bar{T}_\nu^{(n)}} - \frac{\lambda^2}{2}\left(\frac{Var_\omega \bar{T}_\nu^{(n)}}{(E_\omega \bar{T}_\nu^{(n)})^2}-1\right)}$$

for all $\lambda \geq 0$. Therefore

$$Q\left(\exists \lambda \geq 0 : \phi_{1,n}(\lambda) \notin \left[\frac{1-\lambda n^{-\varepsilon/s}}{1+\lambda}, \frac{1}{1+\lambda-(\lambda+3\lambda^2/2)\,n^{-\varepsilon/s}}\right],\ M_1 > n^{(1-\varepsilon)/s}\right)$$
$$\leq Q\left(\frac{E_\omega S}{E_\omega \bar{T}_\nu^{(n)}} \geq n^{-\varepsilon/s}, \quad M_1 \geq n^{(1-\varepsilon)/s}\right) + Q\left(\frac{Var_\omega \bar{T}_\nu^{(n)}}{(E_\omega \bar{T}_\nu^{(n)})^2} - 1 \geq n^{-\varepsilon/s}, \quad M_1 \geq n^{(1-\varepsilon)/s}\right)$$

Now, since $E_\omega \bar{T}_\nu^{(n)} \geq M_1$ we have

$$Q\left(\frac{E_\omega S}{E_\omega \bar{T}_\nu^{(n)}} \geq n^{-\varepsilon/s},\ M_1 \geq n^{(1-\varepsilon)/s}\right) \leq Q\left(E_\omega S \geq n^{(1-2\varepsilon)/s},\ M_1 \geq n^{(1-\varepsilon)/s}\right) = o\left(n^{-(6-8\varepsilon)/5}\right),$$

where the last equality is from Corollary 4.2. Also, by Lemma 4.3 we have

$$Q\left(\frac{Var_\omega \bar{T}_\nu^{(n)}}{(E_\omega \bar{T}_\nu^{(n)})^2} - 1 \geq n^{-\varepsilon/s}, \quad M_1 \geq n^{(1-\varepsilon)/s}\right) = o\left(n^{-2+4\varepsilon}\right).$$

Then, since $-2+4\varepsilon < \frac{-6+8\varepsilon}{5}$ when $\varepsilon < \frac{1}{8}$ the lemma is proved. $\square$

Corollary 4.5. *Let $\varepsilon < \frac{1}{8}$. Then $P-a.s.$, for any sequence $i_k = i_k(\omega)$ such that $i_k \in (n_{k-1}, n_k]$ and $M_{i_k} > d_k^{(1-\varepsilon)/s}$ we have*

$$\lim_{k\to\infty} \phi_{i_k,d_k}(\lambda) = \frac{1}{1+\lambda}, \quad \forall \lambda \geq 0, \tag{60}$$

and thus

$$\lim_{k\to\infty} P_\omega^{\nu_{i_k-1}}\left(\bar{T}_{\nu_{i_k}}^{(d_k)} > x\mu_{i_k,d_k,\omega}\right) = \Psi(x), \quad \forall x \in \mathbb{R}. \tag{61}$$

Proof. For $i \in (n_{k-1}, n_k]$ and all k large enough $\phi_{i,d_k}(\lambda)$ only depends on the environment to the right of zero, and thus has the same distribution under P and Q. Therefore, Lemma 4.4 gives that there exists an $\varepsilon' > 0$ such that

$$P\left(\exists i \in (n_{k-1}, n_k], \lambda \geq 0 : \phi_{i,d_k}(\lambda) \notin \left[\frac{1-\lambda d_k^{-\varepsilon/s}}{1+\lambda}, \frac{1}{1+\lambda-\left(\lambda+\frac{3\lambda^2}{2}\right)d_k^{-\varepsilon/s}}\right],\ M_i > d_k^{(1-\varepsilon)/s}\right)$$
$$\leq d_k Q\left(\exists \lambda \geq 0 : \phi_{1,d_k}(\lambda) \notin \left[\frac{1-\lambda d_k^{-\varepsilon/s}}{1+\lambda}, \frac{1}{1+\lambda-\left(\lambda+\frac{3\lambda^2}{2}\right)d_k^{-\varepsilon/s}}\right],\ M_1 > d_k^{(1-\varepsilon)/s}\right)$$
$$= o\left(d_k^{-\varepsilon'}\right).$$

Since this last term is summable in k, the Borel-Cantelli Lemma gives that $P-a.s.$ there exists a $k_0 = k_0(\omega)$ such that for all $k \geq k_0$ we have

$$i \in (n_{k-1}, n_k] \text{ and } M_i \geq d_k^{(1-\varepsilon)/s} \Rightarrow \phi_{i,d_k}(\lambda) \in \left[\frac{1-\lambda d_k^{-\varepsilon/s}}{1+\lambda}, \frac{1}{1+\lambda-\left(\lambda+\frac{3\lambda^2}{2}\right)d_k^{-\varepsilon/s}}\right] \quad \forall \lambda \geq 0,$$

which proves (60). Then, (61) follows immediately because $\frac{1}{1+\lambda}$ is the Laplace transform of an exponential disribution. $\square$

4.2 Quenched Exponential Limits Along a Subsequence

In the previous subsection we showed that the time to cross a single large block is approximately exponential. In this section we show that there are subsequences in the environment where the crossing time of a single block dominates the crossing times of all the other blocks. In this case the crossing time of all the blocks is approximately exponentially distributed. Recall the definition of M_i in (16). For any integer $n \geq 1$, and constants $C > 1$ and $\eta > 0$, define the event

$$\mathcal{D}_{n,C,\eta} := \left\{ \exists i \in [1, \eta n] : M_i^2 \geq C \sum_{j: i \neq j \leq n} \sigma_{j,n,\omega}^2 \right\}$$

Lemma 4.6. *Assume $s < 2$. Then for any $C > 1$ and $\eta > 0$ we have* $\liminf_{n\to\infty} Q\left(\mathcal{D}_{n,C,\eta}\right) > 0$*.*

Proof. First, note that since $\sigma_{i,n,\omega}^2 \geq M_i^2$ and $C > 1$ we have

$$Q\left(\mathcal{D}_{n,C,\eta}\right) = \sum_{i=1}^{\eta n} Q\left(M_i^2 \geq C \sum_{j: i\neq j \leq n} \sigma_{j,n,\omega}^2 \right). \tag{62}$$

Thus, we want to get a lower bound on $Q\left(M_i^2 \geq C \sum_{j:i\neq j\leq n} \sigma_{j,n,\omega}^2\right)$ that is uniform in i. The following formula for the quenched variance of $\bar{T}_\nu^{(n)}$ can be deduced from (15) by setting $\rho_{\nu_{-b_n}=0}$:

$$\begin{aligned}
Var_\omega \bar{T}_\nu^{(n)} &= 4\sum_{j=0}^{\nu-1}(W_{\nu_{-b_n}+1,j} + W_{\nu_{-b_n}+1,j}^2) + 8\sum_{j=0}^{\nu-1}\sum_{i=\nu_{-b_n}+1}^{j} \Pi_{i+1,j}(W_{\nu_{-b_n}+1,i} + W_{\nu_{-b_n}+1,i}^2) \\
&\leq 4\sum_{j=0}^{\nu-1}(W_{\nu_{-b_n}+1,j} + W_{\nu_{-b_n}+1,j}^2) + 8\sum_{j=0}^{\nu-1}\sum_{i=\nu_{-b_n}+1}^{j} W_{\nu_{-b_n}+1,j}(1 + W_{\nu_{-b_n}+1,i}) \\
&\leq 4\sum_{j=0}^{\nu-1}(W_{\nu_{-b_n}+1,j} + W_{\nu_{-b_n}+1,j}^2) + 8\left(\sum_{j=0}^{\nu-1} W_{\nu_{-b_n}+1,j}\right)\left(\sum_{i=\nu_{-b_n}+1}^{\nu-1}(1 + W_{\nu_{-b_n}+1,i})\right),
\end{aligned}$$

where the first inequality is because $W_{\nu_{-b_n}+1,j} = W_{i+1,j} + \Pi_{i+1,j}W_{\nu_{-b_n}+1,i}$. Next, note that if $\nu_{k-1} \leq j < \nu_k$ for some $k > -b_n$, then

$$W_{\nu_{-b_n}+1,j} = \sum_{l=\nu_{-b_n}+1}^{j} \Pi_{l,j} = \sum_{l=\nu_{-b_n}+1}^{\nu_{k-1}-1} \Pi_{l,\nu_{k-1}-1}\Pi_{\nu_{k-1},j} + \sum_{l=\nu_{k-1}}^{j} \Pi_{l,j} \leq (\nu_k - \nu_{-b_n})M_k,$$

where the last inequality is because, under Q, $\Pi_{l,\nu_{k-1}-1} < 1$ for all $l < \nu_{k-1}$. Therefore,

$$
\begin{aligned}
Var_\omega \bar{T}_\nu^{(n)} \le 4\nu_1 \left((\nu_1-\nu_{-b_n})M_1 + (\nu_1-\nu_{-b_n})^2 M_1^2\right) \\
+ 8\left(\nu_1(\nu_1-\nu_{-b_n})M_1\right)\left((\nu_1-\nu_{-b_n}) + \sum_{i=-b_n+1}^{1}(\nu_k-\nu_{k-1})(\nu_k-\nu_{-b_n})M_k\right) \\
\le (\nu_1-\nu_{-b_n})^4\left(12M_1 + 4M_1^2 + 8M_1\sum_{k=-b_n+1}^{1} M_k\right).
\end{aligned}
$$

Similarly, we have that $\sigma_{j,n,\omega}^2 \le (\nu_j - \nu_{j-1-b_n})^4 \left(12M_j + 4M_j^2 + 8M_j \sum_{k=j-b_n}^{j} M_k\right)$ $Q-a.s.$ for any j. Now, define the events

$$
F_n := \bigcap_{j\in(-b_n,n]} \{\nu_j - \nu_{j-1} \le b_n\}, \quad \text{and} \quad G_{i,n,\varepsilon} := \bigcap_{j\in[i-b_n,i+b_n]\setminus\{i\}} \left\{M_j \le n^{(1-\varepsilon)/s}\right\} \tag{63}
$$

Then, on the event $F_n \cap G_{i,n,\varepsilon} \cap \left\{M_i \le 2n^{1/s}\right\}$ we have for $j \in (i, i+b_n]$ that

$$
\begin{aligned}
\sigma_{j,n,\omega}^2 &\le b_n^4(b_n+1)^4\left(12n^{(1-\varepsilon)/s} + 4n^{(2-2\varepsilon)/s} + 8n^{(1-\varepsilon)/s}(b_n n^{(1-\varepsilon)/s} + 2n^{1/s})\right) \\
&\le b_n^5(b_n+1)^4\left(12n^{(1-\varepsilon)/s} + 12n^{(2-2\varepsilon)/s} + 16n^{(2-\varepsilon)/s}\right) \le 80 b_n^9 n^{(2-\varepsilon)/s},
\end{aligned}
$$

where the last inequality holds for all n large enough. Therefore, for all n large enough

$$
\begin{aligned}
Q\left(M_i^2 \ge C\sum_{j:i\ne j\le n}\sigma_{j,n,\omega}^2\right) &\ge Q\left(4n^{2/s} \ge M_i^2 \ge C\sum_{j:i\ne j\le n}\sigma_{j,n,\omega}^2,\ F_n,\ G_{i,n,\varepsilon}\right) \\
&\ge Q\left(4n^{2/s} \ge M_i^2 \ge C\left(\sum_{j\in[1,n]\setminus[i,i+b_n]}\sigma_{j,n,\omega}^2 + 80b_n^9 n^{(2-\varepsilon)/s}\right),\ F_n,\ G_{i,n,\varepsilon}\right) \\
&\ge Q\left(M_i \in [n^{1/s}, 2n^{1/s}],\ \nu_i - \nu_{i-1} \le b_n\right) \\
&\quad\times Q\left(\sum_{j\in[1,n]\setminus[i,i+b_n]}\sigma_{j,n,\omega}^2 + 80b_n^9 n^{(2-\varepsilon)/s} \le \frac{n^{2/s}}{C},\ \tilde{F}_n,\ G_{i,n,\varepsilon}\right),
\end{aligned}
$$

where $\tilde{F}_n := F_n \setminus \{\nu_i - \nu_{i-1} \le b_n\}$. Note that in the last inequality we used that $\sigma_{j,n,\omega}^2$ is independent of M_i for $j \notin [i, i+b_n]$. Also, note that we can replace $\tilde{F}_n$ by F_n in the last line above because it will only make the probability smaller. Then, since $\sum_{j\in[1,n]\setminus[i,i+b_n]}\sigma_{j,n,\omega}^2 \le Var_\omega T_{\nu_n}$ we have

$$
\begin{aligned}
&Q\left(M_i^2 \ge C\sum_{j:i\ne j\le n}\sigma_{j,n,\omega}^2\right) \\
&\quad\ge Q\left(M_1 \in [n^{1/s}, 2n^{1/s}],\ \nu \le b_n\right) Q\left(Var_\omega T_{\nu_n} \le n^{2/s}C^{-1} - 40b_n^7 n^{(2-\varepsilon)/s},\ F_n,\ G_{i,n,\varepsilon}\right) \\
&\quad\ge \left(Q(M_1 \in [n^{1/s}, 2n^{1/s}]) - Q(\nu > b_n)\right) \\
&\qquad\times\left(Q\left(Var_\omega T_{\nu_n} \le n^{2/s}(C^{-1} - 40b_n^7 n^{-\varepsilon/s})\right) - Q(F_n^c) - Q(G_{i,n,\omega}^c)\right) \\
&\quad\sim C_3(1-2^{-s})\frac{1}{n}L_{\frac{s}{2},b}\left(C^{-1}\right),
\end{aligned} \tag{64}
$$

where the asymptotics in the last line are from (13), (17), and Theorem 1.3, as well as the fact that $Q(F_n^c) + Q(G_{i,n,\omega}^c) \le (n+b_n)Q(\nu > b_n) + 2b_n Q(M_1 > n^{(1-\varepsilon)/s}) = \mathcal{O}\left(ne^{-C_2 b_n}\right) + o(n^{-1+2\varepsilon})$ due to (13) and (17). Combining (62) and (64) finishes the proof. $\square$

Corollary 4.7. *Assume $s < 2$. Then for any $\eta \in (0,1)$, $P-a.s.$ there exists a subsequence $n_{k_m} = n_{k_m}(\omega,\eta)$ of $n_k = 2^{2^k}$ such that for α_m, β_m, and γ_m defined as in* (32) *we have that*

$$\exists i_m = i_m(\omega,\eta) \in (\alpha_m, \beta_m]: \quad M_{i_m}^2 \ge m \sum_{j \in (\alpha_m, \gamma_m] \backslash \{i_m\}} \sigma^2_{j, d_{k_m}, \omega}\,. \tag{65}$$

Proof. Define the events

$$\mathcal{D}'_{k,C,\eta} := \left\{ \exists i \in (n_{k-1}, n_{k-1} + \eta d_k] : M_i^2 \ge C \sum_{j \in (n_{k-1}, n_k] \backslash \{i\}} \sigma^2_{j,d_k,\omega} \right\}.$$

Note that since Q is invariant under shifts of the ν_i, $Q(\mathcal{D}'_{k,C,\eta}) = Q(\mathcal{D}_{d_k,C,\eta})$. Also, due to the reflections of the random walk the event $\mathcal{D}'_{k,C,\eta}$ only depends on the environment between $\nu_{n_{k-1}-b_{d_k}}$ and ν_{n_k}. Thus, for k large enough $\mathcal{D}'_{k,C,\eta}$ only depends on the environment to the right of zero and therefore $P(\mathcal{D}'_{k,C,\eta}) = Q(\mathcal{D}'_{k,C,\eta}) = Q(\mathcal{D}_{d_k,C,\eta})$. Therefore $\liminf_{k\to\infty} P(\mathcal{D}'_{k,C,\eta}) > 0$. Also, since $n_{k-1} - b_{d_k} > n_{k-2}$ for all $k \ge 4$, we have that $\{\mathcal{D}'_{2k,C,\eta}\}_{k=2}^\infty$ is an independent sequence of events. Thus, we get that for any $C > 1$ and $\eta \in (0,1)$, infinitely many of the events $\mathcal{D}_{k,C,\eta}$ occur $P-a.s.$ Therefore, $P-a.s.$ there is a subsequence $k_m = k_m(\omega)$ such that $\omega \in \mathcal{D}_{k_m,m,\eta}$ for all m. In particular, for this subsequence k_m we have that (65) holds. $\square$

Theorem 4.8. *Assume $s < 2$. Then for any $\eta \in (0,1)$, $P-a.s.$ there exists a subsequence $n_{k_m} = n_{k_m}(\omega,\eta)$ of $n_k = 2^{2^k}$ such that for α_m, β_m and γ_m defined as in* (32) *and any sequence $x_m \in (\nu_{\beta_m}, \nu_{\gamma_m}]$ we have*

$$\lim_{m\to\infty} P_\omega^{\nu_{\alpha_m}} \left(\frac{\bar{T}_{x_m}^{(d_{k_m})} - E_\omega^{\nu_{\alpha_m}} \bar{T}_{x_m}^{(d_{k_m})}}{\sqrt{v_{k_m,\omega}}} \le x \right) = \Psi(x+1), \quad \forall x \in \mathbb{R}.$$

Proof. First, note that $P\left(\max_{j\in(n_{k-1},n_k]} M_j \le d_k^{(1-\varepsilon)/s}\right) = \left(1 - P\left(M_1 > d_k^{(1-\varepsilon)/s}\right)\right)^{d_k} = o\left(e^{-d_k^{\varepsilon/2}}\right)$, where the last equality is due to (17). Therefore, the Borel-Cantelli Lemma gives that $P-a.s.$ we have

$$\max_{j\in(n_{k-1},n_k]} M_j > d_k^{(1-\varepsilon)/s} \quad \text{for all } k \text{ large enough.} \tag{66}$$

Therefore, $P-a.s.$ we may assume that (66) holds, the conclusion of Corollary 4.5 holds, and that there exist subsequences $n_{k_m} = n_{k_m}(\omega,\eta)$ and $i_m = i_m(\omega,\eta)$ as specified in Corollary 4.7. Then, by the choice of our subsequence n_{k_m}, only the crossing of the largest block (i.e. from ν_{i_m-1} to ν_{i_m}) is relevant in the limiting distribution. Indeed,

$$P_\omega^{\nu_{\alpha_m}} \left(\left| \frac{\left(\bar{T}_{\nu_{i_m-1}}^{(d_{k_m})} - E_\omega^{\nu_{\alpha_m}} \bar{T}_{\nu_{i_m-1}}^{(d_{k_m})}\right) + \left(\bar{T}_{x_m}^{(d_{k_m})} - \bar{T}_{\nu_{i_m}}^{(d_{k_m})} - E_\omega^{\nu_{i_m}} \bar{T}_{x_m}^{(d_{k_m})}\right)}{\sqrt{v_{k_m,\omega}}} \right| \ge \varepsilon \right)$$
$$\le \frac{Var_\omega\left(\bar{T}_{x_m}^{(d_{k_m})} - \bar{T}_{\nu_{\alpha_m}}^{(d_{k_m})}\right) - \sigma^2_{i_m,d_{k_m},\omega}}{\varepsilon^2 v_{k_m,\omega}} \le \frac{\sum_{j\in(\alpha_m,\gamma_m]\backslash\{i_m\}} \sigma^2_{j,d_{k_m},\omega}}{\varepsilon^2 M_{i_m}^2} \le \frac{1}{\varepsilon^2 m},$$

where in the second to last inequality we used that $v_{k_m,\omega} \geq \sigma^2_{i_m,d_{k_m},\omega} \geq M^2_{i_m}$, and the last inequality is due to our choice of the sequence i_m. Thus we have reduced the proof of the Theorem to showing that

$$\lim_{m\to\infty} P_\omega^{\nu_{i_m-1}} \left(\frac{\bar{T}_{\nu_{i_m}}^{(d_{k_m})} - \mu_{i_m,d_{k_m},\omega}}{\sqrt{v_{k_m,\omega}}} \leq x \right) = \Psi(x+1), \quad \forall x \in \mathbb{R}. \tag{67}$$

Now, since i_m is chosen so that $M_{i_m} = \max_{j\in(n_{k_m-1},n_{k_m}]} M_j$, we have that $M_{i_m} \geq d_{k_m}^{(1-\varepsilon)/s}$ for any $\varepsilon > 0$ and all m large enough. Then, the conclusion of Corollary 4.5 gives that

$$\lim_{m\to\infty} P_\omega^{\nu_{i_m-1}} \left(\frac{\bar{T}_{\nu_{i_m}}^{(d_{k_m})}}{\mu_{i_m,d_{k_m},\omega}} \leq x \right) = \Psi(x).$$

Thus, the proof will be complete if we can show

$$\lim_{m\to\infty} \frac{\mu_{i_m,d_{k_m},\omega}}{\sqrt{v_{k_m,\omega}}} = 1. \tag{68}$$

However, by our choice of n_{k_m} and i_m we have

$$\sigma^2_{i_m,d_{k_m},\omega} \geq M^2_{i_m} \geq m \sum_{j\in(\alpha_m,\gamma_m]\backslash\{i_m\}} \sigma^2_{j,d_{k_m}\omega} = m\left(v_{k_m,\omega} - \sigma^2_{i_m,d_{k_m},\omega}\right),$$

which implies that

$$1 \leq \frac{v_{k_m,\omega}}{\sigma^2_{i_m,d_{k_m},\omega}} \leq \frac{m+1}{m} \underset{m\to\infty}{\longrightarrow} 1. \tag{69}$$

Also, we can use Lemma 4.3 to show that for k large enough and $\varepsilon > 0$

$$\begin{aligned} &P\left(\exists i \in (n_{k-1}, n_k] : \left| \frac{\sigma^2_{i,d_k,\omega}}{\mu^2_{i,d_k,\omega}} - 1 \right| \geq d_k^{-\varepsilon/s}, \quad M_i \geq d_k^{(1-\varepsilon)/s} \right) \\ &\leq d_k Q\left(\left| \frac{Var_\omega \bar{T}_\nu^{(d_k)}}{\left(E_\omega \bar{T}_\nu^{(d_k)}\right)^2} - 1 \right| \geq d_k^{-\varepsilon/s}, \quad M_1 \geq d_k^{(1-\varepsilon)/s} \right) = o\left(d_k^{-1+4\varepsilon}\right). \end{aligned}$$

Then, for $\varepsilon < \frac{1}{4}$ the Borel-Cantelli Lemma gives that $P-a.s.$ there exists a $k_0 = k_0(\omega)$ such that for $k \geq k_0$ and $i \in (n_{k-1}, n_k]$ with $M_i \geq d_k^{(1-\varepsilon)/s}$ we have $\left| \frac{\sigma^2_{i,d_k,\omega}}{\mu^2_{i,d_k,\omega}} - 1 \right| < d_k^{-\varepsilon/s}$. In particular, since $M_{i_m} \geq d_{k_m}^{(1-\varepsilon)/s}$ for all m large enough, we have that

$$\lim_{m\to\infty} \frac{\sigma^2_{i_m,d_{k_m},\omega}}{\mu^2_{i_m,d_{k_m},\omega}} = 1. \tag{70}$$

Since (69) and (70) imply (68), the proof is complete. □

Proof of Theorem 1.2:
As in the proof of Theorem 1.1 this follows from Proposition 1.4. □

5 Stable Behavior of the Quenched Variance

Recall from Theorem 1.5 that $Q\left(Var_\omega T_\nu > x\right) \sim K_\infty x^{-s/2}$. Since the sequence of random variables $\left\{Var_\omega(T_{\nu_i} - T_{\nu_{i-1}})\right\}_{i\in\mathbb{N}}$ is stationary under Q (and weakly dependent) it is somewhat natural to expect that $n^{-2/s}Var_\omega T_{\nu_n}$ converges in distribution (under Q) to a stable law of index $\frac{s}{2} < 1$.

Proof of Theorem 1.3:
Obviously it is enough to prove that the second equality in (9) holds and that

$$\lim_{n\to\infty} Q\left(\left|Var_\omega T_{\nu_n} - \sum_{i=1}^n (E_\omega^{\nu_{i-1}} T_{\nu_i})^2\right| > \delta n^{2/s}\right) = 0, \quad \forall \delta > 0. \tag{71}$$

Note that (71) is the analogue of [9, Corollary 5.6] for the random walk without added reflections. While Lemma 5.5 and Corollary 5.6 in [9] were both proved for the random walk modified with reflections, the proofs hold unaltered for the random walk without reflections. Thus it is enough to prove the second equality in (9). To this end, first note that

$$\frac{1}{n^{2/s}}\sum_{i=1}^n \left(E_\omega^{\nu_{i-1}} T_{\nu_i}\right)^2 = \frac{1}{n^{2/s}}\sum_{i=1}^n \left(\left(E_\omega^{\nu_{i-1}} T_{\nu_i}\right)^2 - \left(E_\omega^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)}\right)^2\right) \tag{72}$$

$$+ \frac{1}{n^{2/s}}\sum_{i=1}^n \left(E_\omega^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)}\right)^2 \mathbf{1}_{M_i \le n^{(1-\varepsilon)/s}} \tag{73}$$

$$+ \frac{1}{n^{2/s}}\sum_{i=1}^n \left(E_\omega^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)}\right)^2 \mathbf{1}_{M_i > n^{(1-\varepsilon)/s}}. \tag{74}$$

Therefore, it is enough to show that (72) and (73) converge to 0 in distribution (under Q) and that

$$\lim_{n\to\infty} Q\left(\frac{1}{n^{2/s}}\sum_{i=1}^n \left(E_\omega^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)}\right)^2 \mathbf{1}_{M_i > n^{(1-\varepsilon)/s}} \le x\right) = L_{\frac{s}{2},b}(x) \tag{75}$$

for some $b > 0$. To prove that (72) converges to 0 in distribution, first note that factoring gives

$$\left(E_\omega^{\nu_{i-1}} T_{\nu_i}\right)^2 - \left(E_\omega^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)}\right)^2 \le 2E_\omega^{\nu_{i-1}} T_{\nu_i}\left(E_\omega^{\nu_{i-1}} T_{\nu_i} - E_\omega^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)}\right).$$

Therefore, for any $\delta > 0$

$$\begin{aligned} &Q\left(\sum_{i=1}^n \left(\left(E_\omega^{\nu_{i-1}} T_{\nu_i}\right)^2 - \left(E_\omega^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)}\right)^2\right) > \delta n^{2/s}\right) \\ &\quad \le Q\left(\sum_{i=1}^n 2E_\omega^{\nu_{i-1}} T_{\nu_i}\left(E_\omega^{\nu_{i-1}} T_{\nu_i} - E_\omega^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)}\right) > \delta n^{2/s}\right) \\ &\quad \le nQ\left(E_\omega T_\nu - E_\omega \bar{T}_\nu^{(n)} > 1\right) + Q\left(2E_\omega T_{\nu_n} > \delta n^{2/s}\right). \end{aligned} \tag{76}$$

Then, [9, Lemma 3.2 & Theorem 1.1] give that both terms in (76) tend to zero as $n \to \infty$. The proof that (73) converges in distribution to 0 is essentially a counting argument. Since the M_i are all independent and from (17) we know the asymptotics of $Q(M_i > x)$, we can get good bounds on the number of $i \leq n$ with $M_i \in (n^\alpha, n^\beta]$. Then, since by [9, (15)] we have $Q\left(E_\omega^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)} \geq n^\beta, M_i \leq n^\alpha\right) = o\left(e^{-n^{(\beta-\alpha)/5}}\right)$ we can also get good bounds on the number of $i \leq n$ with $E_\omega^{\nu_{i-1}} \bar{T}_{\nu_i}^{(n)} \in (n^\alpha, n^\beta]$. The details of this argument are essentially the same as the proof of Lemma 5.5 in [9] and will thus be ommitted. Finally, we will use [7, Theorem 5.1(III)] to prove (75). Now, Theorem 1.5 gives that $Q\left((E_\omega T_\nu)^2 \mathbf{1}_{M_1 > n^{(1-\varepsilon)/s}} > xn^{2/s}\right) \sim K_\infty x^{-s/2} n^{-1}$, and [9, Lemma 3.4] gives bounds on the mixing of the array $\left\{\left(E_\omega^{\nu_{i-1}} T_{\nu_i}\right)^2 \mathbf{1}_{M_i > n^{(1-\varepsilon)/s}}\right\}_{i \in \mathbb{Z}, n \in \mathbb{N}}$. This is enough to verify the first two conditions of [7, Theorem 5.1(III)]. The final condition that needs to be verified is

$$\lim_{\delta \to 0} \limsup_{n \to \infty} n E_Q \left[n^{-2/s} (E_\omega \bar{T}_\nu^{(n)})^2 \mathbf{1}_{M_1 > n^{(1-\varepsilon)/s}} \mathbf{1}_{n^{-1/s} E_\omega \bar{T}_\nu^{(n)} \leq \delta} \right] = 0\,. \tag{77}$$

By Theorem 1.5 we have that there exists a constant $C_4 > 0$ such that for any $x > 0$,

$$Q\left(E_\omega \bar{T}_\nu^{(n)} > xn^{1/s}, M_1 > n^{(1-\varepsilon)/s}\right) \leq Q\left(E_\omega T_\nu > xn^{1/s}\right) \leq C_4 x^{-s} \frac{1}{n}.$$

Then using this we have

$$\begin{aligned} & n E_Q \left[n^{-2/s} \left(E_\omega \bar{T}_\nu^{(n)}\right)^2 \mathbf{1}_{M_1 > n^{(1-\varepsilon)/s}} \mathbf{1}_{n^{-1/s} E_\omega \bar{T}_\nu^{(n)} \leq \delta} \right] \\ & = n \int_0^{\delta^2} Q\left(\left(E_\omega \bar{T}_\nu^{(n)}\right)^2 > xn^{2/s}, M_1 > n^{(1-\varepsilon)/s} \right) dx \\ & \leq C_4 \int_0^{\delta^2} x^{-s/2} dx = \frac{C_4 \delta^{2-s}}{1 - s/2}\,, \end{aligned}$$

where the last integral is finite since $s < 2$. (77) follows, and therefore by [7, Theorem 5.1(III)] we have that (75) holds. □

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