Travelling Wave Solutions of the Heat Equation in an unbounded Cylinder with a Non-linear Boundary Condition.

Overview

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Consider the heat equation in an unbounded cylinder $\mathbb{R} \times \Omega$ with a non-linear dissipation condition on the boundary

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R} \times \Omega, \\ \dfrac{\partial u}{\partial n} = f(u) & \text{on } \mathbb{R}^+ \times \mathbb{R} \times \partial\Omega. \end{cases} \tag{1}$$

This problem arises in the study of transient boiling processes, where non-linearities f having the characteristics of a boiling-curve (see figure 1) are of special interest. One phenomenon, which is subject to a lot of research activities in the field of transient boiling processes, is the existence of so-called heat-waves. These are travelling wave solutions

$$(t, x, y) \to u(x + ct, y) \quad , \ (t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \tag{2}$$

of (1). Finding such solutions amounts to solving the elliptic equation

$$\begin{cases} \Delta u - c\,\partial_x u = 0 & \text{in } \mathbb{R} \times \Omega, \\ \dfrac{\partial u}{\partial n} = f(u) & \text{on } \mathbb{R} \times \partial\Omega. \end{cases} \tag{3}$$

Our main result is the existence of a non-trivial solution of (3) for a large class of non-linearities f, including those with the characteristics of boiling curves. We use a variational approach. More specifically, we find solutions as minimizers of the functional

$$\mathcal{E}(u) := \frac{1}{2} \int\limits_{\mathbb{R}\times\Omega} |\mathrm{D}u|^2 \, \mathrm{e}^{-x} \, \mathrm{d}(x, y) \tag{4}$$

in the class of functions in the weighted Sobolev space $\mathrm{H}^1_2(\mathbb{R} \times \Omega, \mathrm{e}^{-x})$ satisfying the constraint

$$\int\limits_{\mathbb{R}\times\partial\Omega} F(u)\, \mathrm{e}^{-x} \, \mathrm{d}S(y)\mathrm{d}x \ = \ 1 \quad \text{where} \quad F(u) := \int\limits_0^u f(t)dt. \tag{5}$$

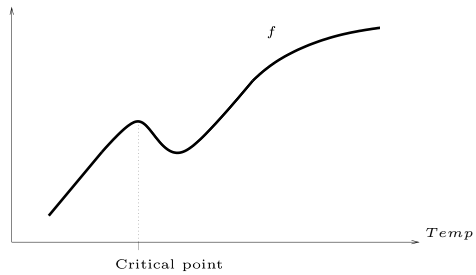

Figure 1: Boiling Curve

The associated Lagrange multiplier of a minimizer will turn out to be the propagation speed c of the travelling wave.

The main obstacle in the minimization problem above is the lack of compactness caused by the unboundedness of the domain $\mathbb{R} \times \Omega$. We overcome this obstacle by constructing a special minimizing sequence with sufficient decay properties in order to find a minimizer as the weak limit hereof.

Travelling wave solutions of semilinear reaction diffusion equations in unbounded cylinders have been studied by a number of authors including Aronson, Weinberger, Berestycki, Larrouturou, Nirenberg, P.L. Lions and many others. None of the existing results, however, cover the nonlinear boundary condition in (3). Most notably, the boundary condition $\frac{\partial u}{\partial n} = f(u)$ complicates the use of maximum principles, which play a central role in all of the existing results.

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