HW 13 SOLUTIONS

The Way of Analysis

p. 294:

1.) Show that an analytic function f is determined by its values in any neighborhood:

SOLUTION: (I wrote this out in far too much detail, I'm afraid. It's much shorter and quicker if you have a pencil and paper in front of you so you can draw a little diagram...)

First, it's important to read this problem carefully and make sure you are comfortable with all the definitions involved, because it's important that we realize there is something to actually prove! We need to show that the value of f at any point $y \in (a, b)$ is completely determined by the values of f in the neighborhood of any other point $p \in (a, b)$.

It's also important to realize (and most of you probably already do) that if we know the values of f on a neighborhood of p, then we know the power series of f centered at the point p. (This is because knowing the values of f on an interval around p allows us to compute the derivatives $f^{(n)}(p)$ which is all we need to do to write down the power series.

One more preliminary remark: obviously, the power series at a point p determines the values of f in the interval of convergence.

Now we start the solution proper: Use the notation established in the preliminary remarks above. Suppose that we have a point p_1 in the interval I around p, where I is the interval of convergence of the power series centered at p. Consider now the power series for f centered at p_1. Clearly, the the first power series (centered at p) determines the second (using the formula for the a_n's in terms of the derivatives of f at p_1). This means that the first power series determines the values of f on the entire interval of convergence of this second power series. Continuing in this way, we can connect p and y by laying down a finite number of intervals.

There is one thing we need to do in order to justify that last statement: the point of this problem is to show that we can repeat this a finite number of times to connect up any two points p and y. It is obvious that we can do this IF we can show that there is a lower bound for the length of any intervals of convergence for power series with centers in $[x, y]$. Let's prove that such a lower bound exists by contradiction: suppose (for contradiction) that there is a sequence of points $x_n \in [x, y]$ with the radius of convergence at x_n going to zero. By compactness, a subsequence converges to a point $z \in [x, y]$. However, since f is analytic, the power series at z converges on some positive radius R. This gives the contradiction: as x_n goes to z, its radius of convergence has get closer and closer to at least R — but its radius of convergence is supposed to be going to zero. (To make this last bit of the argument more explicit: you could say that there is an M so $m \geq M$ implies $|x_m - z| < R/4$. Then for such M, by a Theorem we proved in class (and which is in the reading) you must have $m \geq M$ implies that the power series centered at x_m has radius of convergence at least $R/2$. (Now a picture would be very helpful....)

2.) If f is analytic near 0 and $f(0) = 0$, show that $f(x)/x$ is also analytic near 0:

We can write $f = \sum_0^\infty a_n\, x^n$ near 0 since it is analytic; the radius of convergence R is positive. Furthermore, we know that $a_0 = f(0) = 0$. We should have that

$$f/x = \sum_1^\infty a_n\, x^{n-1}\,.$$

We just need to make sure that this converges on some open interval about 0. But this is clear since the radius of convergence of this power series is (notice that a_n is now no longer the coefficient of x^n)

$$\frac{1}{R_{\text{new}}} \quad = \quad \limsup_{n\to\infty} |a_n|^{\frac{1}{n-1}}$$

We now want to check that R_{new} is the same as the original radius of convergence, by computing the limit on the right hand side. Writing

$$|a_n|^{\frac{1}{n-1}} = |a_n|^{\frac{1}{n}\cdot\frac{n}{n-1}}$$

and argue that the limsup of the right hand side is $\frac{1}{R}$, where R is the original radius of convergence.

To do that rigorously would be a pain, but it's pretty clear intuitively. Here is another way to finish off the problem: Fix any positive $r < R$. We know that $|a_n|\, r^n < M$ for some constant M (Why??). Now consider any x with $|x| < r$; we have

$$|a_n\, x^{n-1}| = |a_n\, r^n|\, \frac{|x|^n}{r^n}\, |x|^{-1} \leq M\, |x|^{-1}\, (|x|/r)^n\,.$$

The right hand side gives a convergent geometric series, so the series for f/x converges on an interval of radius at least r.

6.) If f is analytic on (a, b), show that so is $\int_c^x f(t)\, dt$:

Fix some point x_0; We will show that there is a power series for the function $\int_c^x f(t)\, dt$ near x_0. This follows from two facts (both follow quite quickly from what we've done in class and from what is in the text):

- $\int_{x_0}^x f(t)\, dt$ is an analytic function of x near x_0 (this uses our theorem that you can integrate power series term by term inside their radius of convergence).
- $\int_c^x f(t)\, dt = [\int_c^{x_0} f(t)\, dt] + [\int_{x_0}^x f(t)\, dt]$ means that the function we are interesting in showing is analytic is the sum of a constant and a power series — so just add the constant in with the a_0 term of the series to get the power series for the function $\int_c^x f(t)\, dt$

7.) Find power series expansions about 0 for $x^2/(1-x^2)$, $1/(1-x)^2$, and $\sqrt{1+x}$:

We work out here one – say $x^2/(1-x^2)$. We know that the geometric series $1+a+a^2+\dots$ sums to $1/(1-a)$. Setting $a = x^2$ gives

$$1/(1-x^2) = 1 + x^2 + x^4 + \dots$$

Multiplying this by x^2 gives

$$x^2/(1-x^2) = x^2 + x^4 + x^6 + \dots$$

8.) Radius of convergence for power series with terms $n^4/(n!)$, $\sqrt{n}$, and $n^2\, 2^n$:

The answers are ∞, one, and $1/2$.

We do only the first one (the hardest, probably):

We know that

$$1/R = \limsup_{n\to\infty} [|a_n|]^{1/n} = \limsup_{n\to\infty} [n^4/(n!)]^{1/n}\,.$$

We will show that this goes to zero. To keep it simple, suppose that n is even. The key point is that

$$n \geq n(n-1)\dots(n/2) \geq (n/2)^{n/2}\,.$$

Hence,

$$[n^4/(n!)]^{1/n} \leq \sqrt{2}\, n^{4/n-1/2}\,.$$

When n is big, this is a negative power of n so it goes to zero as claimed.