MATH 5615H, HW 2 SOLUTIONS

The Way of Analysis
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1.) Let x be a real number, and $\{y_n\}$ be a fixed Cauchy sequence of rational numbers that represents x.

Consider the set $S = \{\{\pm 1, \pm 1, \pm 1, \pm 1, \dots\}\}$, that is, the set of sequences consisting entirely of plus or minus ones in each entry. We know that the set S is uncountable, as it is in one-to-one correspondance with the power set of the natural numbers. (You should make sure you understand why it is in one-to-one correspondance with the power set of the natural numbers, and you should make sure you understand why this makes it an uncountable set.) But for each element $\{a_n\}$ of S, we build a Cauchy sequence $\{m_n\}$ that is equivalent to $\{y_n\}$ as follows: set

$$m_n \equiv y_n + \frac{a_n}{n}.$$

I leave it to you to check that $\{m_n\}$ is Cauchy and equivalent to $\{y_n\}$.

Once you do this, we have shown that there are an uncountable number of sequences equivalent to $\{y_n\}$.

3). If $\{x_j\}$ is a Cauchy sequence consisting entirely of integers, then all of the entries beyond some index are the same integer. We conclude this as follows:

Taking $\epsilon = 1/2$ in the definition of a Cauchy sequence, note that there must be some M so that $m, n \geq M$ implies

$$|x_m - x_n| \leq \frac{1}{2}.$$

But the difference of two integers is always an integer, hence for all $m, n \geq M$, the difference must be zero (this is the only integer less than $\frac{1}{2}$.) This gives us the claim set forward at the start of this solution.

4.) Assume for contradiction that the two sequence are not equivalent. Then there would be an $\epsilon_0 > 0$ so that for arbitrarily large j, $|x_j - y_j| > \epsilon$. (Here, the phrase "arbitrarily large j" means: for all $M > 0$, there exists a $j > M$ so that....) Thus, if $\{z_j\}$ is the shuffled sequence we could find arbitrarily large j satisfying $|z_j - z_{j+1}| > \epsilon$ and so the shuffled sequence could not be a Cauchy sequence.

To prove the opposite direction of the "if and only if": If $\{x_j\}$ and $\{y_j\}$ are equivalent, then given $\epsilon > 0$, there is an M so that $j \geq M$ implies $|x_j - y_j| \leq \epsilon$. Since the two sequences are Cauchy, we can possibly make M larger and still insure that for all $m, n > M$ we have that both of the following are also true:

$$|x_m - x_n| \leq \epsilon. \tag{1}$$

$$|y_m - y_n| \leq \epsilon. \tag{2}$$

Now, suppose that $k, l \geq 2M$, then by definition the terms z_k, z_l of the shuffled sequence are equal to either x_j, x_l or y_j, y_l for some $j, l \geq M$. If both of the z_k, z_l come from the $\{x_j\}$ sequence, or if both come from the $\{y_j\}$ sequence, then we get $|z_k - z_l| \leq \epsilon$ by our choice of M and (1), (2) above. If one comes from the $\{x_j\}$ sequence and the other from the $\{y_j\}$ sequence, then by the triangle inequality we get

$$|z_k - z_l|; = \ |x_j - y_m| \ \leq |x_j - x_m| + |x_m - y_m| \ \leq \epsilon + \epsilon.$$

Thus we have shown that given any ϵ, we can find an M so that if $k, l \geq M$, $|z_k - z_l| \leq 2\epsilon$. This means (since ϵ here was arbitrary) that the shuffled sequence is a Cauchy sequence (when we assumed that the original two sequences are Cauchy and equivalent.)

5). Suppose that $\{x_j\}$ is a Cuachy sequence of rationals and that $\{y_j\}$ is a sequence so that $y_j = x_j$ when $j \geq N$ for some natural number N. We then must show that $\{y_j\}$ is a Cauchy sequence and it is equivalent to $\{x_j\}$.

To show that $\{y_j\}$ is Cauchy, we use the fact that $\{x_j\}$ is Cauchy and say that given $\epsilon > 0$, there is an $\tilde{M}$ so that $m, n \geq \tilde{M}$ implies $|x_m - x_n| \leq \epsilon$. Take then $M = \max(\tilde{M}, N)$. Then for $m, n \geq M$ we have

$$|y_m - y_n| = |x_m - x_n| \leq \epsilon$$

as desired.

It is trivial to show that this sequence of the y_j is equivalent to the x_j: given ϵ, we always pick the same N as defined above and say that if $m \geq N$, then $|x_m - y_m| = |x_m - x_m| = 0 < \epsilon$.

8). It is possible for a Cauchy sequence of positive rational numbers be equivalent to a Cauchy sequence of negative rational numbers. For example,

$$\{\frac{1}{j}\} \tilde{} \{-\frac{1}{j}\}$$

and both of these Cauchy sequences represent the number zero. (To see that the two sequences are equivalent, note that given $\epsilon > 0$, we can pick M so that $\frac{1}{M} < \epsilon/2$. Then for all $m \geq M$, you can check that the two sequences differ at the mth sequence element by at most ϵ.

(Note that it would be impossible to have an example of a positive sequence being equivalent to a negative sequence if the number that the two sequences represented were anything other than 0.)

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1) We want to show that if x, y are real, then $x + y = y + x$. Suppose $x = [\{x_n\}]$ and $y = [\{y_n\}]$. Then

$$\begin{aligned} x + y &= [\{x_n\}] + [\{y_n\}] \\ &= [\{x_n + y_n\}] \quad \text{by definition of addition in the reals,} \\ &= [\{y_n + x_n\}] \quad \text{since addition in rationals is commutative} \\ &= [\{y_n\}] + [\{x_n\}] \quad \text{by definition of addition in the reals} \\ &= y + x. \end{aligned}$$

You prove that addition is associative in a completely similar way - using the fact that addition in the rationals is associative.

3). Let x be a real number and let $\{y_j\}$ be a Cauchy sequence of rationals representing x. We know that given $\frac{1}{n}$, we can find an M so that if $m \geq M$, then $|y_m - x| \leq \frac{1}{2n}$. When

$n = 1$, pick such an M and call it M_1. When $n = 2$, pick such an M and make sure it is bigger than M_1, and call it M_2. Continue in this manner: at the jth stage, pick such an M_j so that for all $m \geq M_j$ we have $|y_m - x| \leq \frac{1}{2j}$.

Now, using the fact that $\{y_j\}$ was Cauchy, you can show that the subsequence $\{y_{M_k}\}$ is also a Cauchy sequence, and it represents the same real number x. Similarly, you can (and should!) show that the sequence defined by $z_k \equiv y_{M_k} - \frac{1}{k}$ is Cauchy, and represents x as well! Finally, it follows (provide the details!) that $z_k < x$ for all k.

5). Let $x < y$ be real numbers. By the previous problem I can find a Cauchy sequence of rationals $y_k < y$ representing y. Note that by choosing a subsequence if necessary, I can be sure that all of the y_k are distinct numbers. (That is, none of them is equal to another one in the sequence.) Given $\frac{1}{N}$, there is an M so that if $k \geq M$, then $|y_k - y| < \frac{1}{N}$. Pick (using the principle of Archimedes) that N so large that $\frac{1}{N} < |x - y|$. Hence for $k \geq M$, we must have $x < y_k < y$. Since we have been careful to reduce to the case where all the y_k are distinct, we have therefore shown that there are an infinite number of rationals (the y_k for $k \geq M$) between $x < y$.

7). Since $x = (x - y) + y$, the triangle inequality gives us

$$\begin{aligned} |x| &= |x - y + y| \\ &\leq |x - y| + |y| \end{aligned}$$

and now if you subtract $|y|$ from both sides of the last inequality you get what we wanted to prove.

10) Suppose $x \in R$ has a Cauchy sequence $\{x_k\}$ of positive rationals. We conclude immediately from Lemma 2.2.5 that $x \geq 0$.

As a corollary, we conclude that if y is represented by a Cauchy sequence $\{y_k\}$ of negative rationals, then (since $x_k - y_k$ is positive), we must have $x - y \geq 0$, or in other words $x \geq y$.

11). We want to prove that no real number satisfies $x^2 = -1$. Obviously $0^2 = 0 \neq -1$. Now notice that $(-x)^2 = (-1)(x)(-1)(x) = (-1)^2x^2 = x^2$. Hence we may assume that $x > 0$. However, the ordered field axioms say that a positive times a positive is still a positive number. But -1 is negative. Hence no such x exists.