6.3.6 $F = x^2 + y^3 + z - 1 = 0$

$DF = [2x,\ 3y^2,\ 1]$

This gives a normal vector to the manifold at each point, since it never vanishes. A two form that orients the manifold is given by

$$det \begin{bmatrix} 2x & | & | \\ 3y^2 & v_1 & v_2 \\ 1 & | & | \end{bmatrix}$$

$2xdy \wedge dz + 3y^2 dx \wedge dz + dx \wedge dy$

6.3.7 $dx \wedge dz \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = 2 > 0$

This is a direct basis.

b. $dx \wedge dz(w_1, w_2) = 14 > 0$

$w_1 = 2v_1 - 3v_2$

$w_2 = v_1 + 2v_2$

$$det \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} = 7 > 0$$

c. $v_1 = \frac{2}{7}w_1 + \frac{3}{7}w_2$ $v_2 = \frac{-1}{7}w_1 + \frac{2}{7}w_2$

$$det \begin{bmatrix} \frac{2}{7} & \frac{-1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix} = \frac{1}{7} > 0$$

6.3.11 $x_1^2 - x_2^2 = x_3$, $2x_1x_2 = x_4$

$$[DF] = \begin{bmatrix} 2x_1 & -2x_2 & -1 & 0 \\ 2x_2 & 2x_1 & 0 & -1 \end{bmatrix}$$

$(4x_1^2 + 4x_2^2)dx_3 \wedge dx_4 + 2x_2dx_2 \wedge dx_4 - 2x_1dx_2 \wedge dx_3 - 2x_2dx_1 \wedge dx_3 + dx_1 \wedge dx_2$

b. To get tangent vectors out of this, we could find the kernel of DF or we could also notice that the equations give us a nice parameterization.

$\eta(x_1, x_2) = (x_1, x_2, x_1^2 - x_2^2, 2x_1x_2)$

$D_1\eta = (1, 0, 2x_1, 2x_2)$

$D_2\eta = (0, 1, -2x_2, 2x_1)$

So we can see that $dx_1 \wedge dx_2(D_1\eta, D_2\eta) = 1$ so this orients the manifold. All other elementary two forms vanish at some point.

6.3.14 $\eta(\theta, \phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$

$D_1\eta = (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0)$ $D_2\eta = (\cos(\theta)\ cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi))$

Note that this is not a basis for T_xM if $\sin(\phi) = 0$

$dy \wedge dz(D_1\eta, D_2\eta) = -\sin^2(\phi)\cos(\theta)$

This equals 0 if $\phi = 0, \pi$ or $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

If $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ then $x = 0$, so the form disappears here.

If $\sin(\phi) = 0, \pi$ then $z = \pm 1$. The tangent space at these points can be given as the span of the basis vectors $span((1, 0, 0), (0, 1, 0))$. We see that $dy \wedge dz((1, 0, 0), (0, 1, 0)) = 0$. It is necessary to check these points this way since our parameterization does not give us the tangent space at these points.

$x^2 + y^2 + z^2 = 1$

$DF = [2x,\ 2y,\ 2z]$

The unit normal is given by $[x, y, z]$ and so the forms agree at the point $(1, 0, 0)$

6.3.15 $x_4 = x_1^2 + x_2^2 + x_3^2$

$Df = [2x_1,\, 2x_2,\, 2x_3,\, -1]$

We can get the tangent vectors out easily by looking at this as a parameterization

$\eta(x_1, x_2, x_3) = (x_1, x_2, x_3, x_1^2 + x_2^2 + x_3^2)$

$D_1\eta = (1, 0, 0, 2x_1)$ $D_2\eta = (0, 1, 0, 2x_2)$ $D_3\eta = (0, 0, 1, 2x_3)$

So $dx_2 \wedge dx_3 \wedge dx_4$ disappears at the origin, which is on the manifold. Therefore this 3-form does not provide an orientation.

$dx_2 \wedge dx_3 \wedge dx_1(D_1\eta, D_2\eta, D_3\eta) = 1$

$dx_2 \wedge dx_1 \wedge dx_3(D_1\eta, D_2\eta, D_3\eta) = -1$

6.4.1 $x^2 + y^2 - z^2 = 0$

$Df = [2x,\, 2y,\, -2z]$

This gives the "outward" normal.

$\omega = 2x dy \wedge dz - 2y dx \wedge dz - 2z dx \wedge dy$

$\eta(r, \theta) = (r\cos(\theta),\, r\sin(\theta),\, r)$

$D_1\eta = (\cos(\theta),\, \sin(\theta),\, 1)$ $D_2\eta = (-r\sin(\theta),\, r\cos(\theta), 0)$

$\omega(D_1\eta, D_2\eta) = -4r^2 < 0$

This parameterization is orientation reversing.

6.4.4 $x_4 = x_1x_2x_3,\ 0 \le x_1, x_2, x_3 \le 1$

$\omega = dx_1 \wedge dx_2 \wedge dx_3$

$\eta(x_1, x_2, x_3) = (x_1, x_2, x_3, x_1x_2x_3)$

$$D\eta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_2x_3 & x_1x_3 & x_1x_2 \end{bmatrix}$$

$\omega(D\eta) = 1 > 0$

So η preserves orientation.

$\int_S x^3 dx_1 \wedge dx_2 \wedge dx_4 = \int (x_3 dx_1 \wedge dx_2 \wedge dx_4)(P_\eta(x)(D\eta) dx_1 dx_2 dx_3 = (\int_0^1 t dt)^3 = \frac{1}{8}$

6.4.8 Let M be an oriented manifold and $\eta : U \to M$ be a parameterization of an open subset of M with U connected. Assume that η is orientation preserving at $x_0 \in U$ with the orientation given by ω. For any other point $x_1 \in U$ there is a path $h : [0, 1] \to M$ which connects x_0, x_1. Consider the function $g : [0, 1] \to \mathbf{R}$, $g(t) = \omega(P_{h(t)}, D\eta)$. We know that $g(0) > 0$ and $g(t) \neq 0$ and g is continuous so by the Intermediate Value Theorem $g(1) > 0$.

6.5.1

1. a,l,j are the same

b,i are the same

d,h,k are the same

c,e,f are the same.

Some of these are different since the are talking about the differential form itself, or the differential form which has been applied to a set of vectors.

6.5.2 $W_{F_1} = x^2dx + (xy)dy - zdz$

$W_{F_2} = x^2dx + xydy + xdz$

$\Phi_{F_1} = x^2(dy \wedge dz) - xy(dx \wedge dz) - z(dx \wedge dy)$

$\Phi_{F_2} = x^2dy \wedge dz - xydx \wedge dz + xdx \wedge dy$

b. $(xy, -y^2)$

$(y, 2, -3x)$

c.

$(3y,\ x^2z, 2z^4)$

$(-x_1^2x_3,\ -x_2x_3, 0)$

6.5.3

a. This is a two form $\Phi(P_x(v_1, v_2))$

b. f should be a vector field $W_{\vec{F}}$

c. ρ is a 3-form.

$\rho(P_x(u, v, w))$

d. $v_1 \circ (v_2 \times v_3)$

e. This is fine.

f. $\Phi_{\vec{F}} = F_1dy \wedge dz - F_2dx \wedge dz + F_3dx \wedge dy$

g. Work is a one form $W_{\vec{F}}(P_x(v_1))$

h. Density takes a function ρ_f

i. Correct as written.