3.6.1

Show that the function

$$f((x)) = x^2 + xy + z^2 - \cos y$$

has a critical point at the origin.

The taylor polynomial at the origin up to order 4 is

$-1 + x^2 + xy + z^2 + 1/2\, y^2 - 1/24\, y^4$

Which has no first order terms, so the origin is a critical point.

The second order terms simplify to the quadratic form

$(x + \frac{1}{2}y)^2 + (\frac{y}{2})^2 + z^2$.

So it is a minimum.

3.6.2

Determine the critical points of

$$f(x, y) = x^3 - 12xy + 8y^3$$

$$Df = [3x^2 - 12y \quad -12x + 24y^2]$$

$Df = \mathbf{0} \to x = 2, \quad y = 1$ or $x = 0, y = 0$

The taylor polynomial around the critical point is

$$-8 + 6\,(x-2)^2 - 12\,(x-2)\,(y-1) + 24\,(y-1)^2 + (x-2)^3 + 8\,(y-1)^3$$

The quadratic terms become

$$6\left[((x-2)-(y-1))^2 + (\sqrt{3}(y-1))^2\right]$$

Which is positive definite, and we have a minimum.

At the point $(x, y) = (0, 0)$ the quadratic form is $-12xy$ which simplifies to

$$-\left(2\sqrt{3}x + \sqrt{3}(y-x)\right)^2 + (\sqrt{3}(y-x))^2.$$

The signature is (1,1) so this is a saddle point.

3.6.3

This mirrors the part of the proof on the positive definite subspace.

Pick $\mathbf{h} \in W$. There exists $C < 0$ such that

$$\frac{f(\mathbf{a} + t\mathbf{h}) - f(\mathbf{a})}{t^2} \geq C|\mathbf{h}|^2 + \frac{rt\mathbf{h}}{t^2}$$

Since $\lim_{t\to 0} \frac{rt\mathbf{h}}{t^2} = 0$ it follows that $f(\mathbf{a} + t\mathbf{h}) < f(\mathbf{a})$ for some $t > 0$ sufficiently small.

3.6.8

$$f(x, y) = \sqrt{1 - x + y^2}$$

The taylor polynomial of degree three is

$$1 - 1/2\,x + 1/2\,y^2 - 1/8\,x^2 + 1/4\,xy^2 - 1/16\,x^3$$

The Taylor polynomial of degree three of

$$f(x, y) + \frac{x}{2}$$

is

$$1 + 1/2\,y^2 - 1/8\,x^2 + 1/4\,xy^2 - 1/16\,x^3$$

Which has no terms of order 1, so the origin is a critical point. The quadratic terms are

$$Q(x, y) = 1/2\,y^2 - 1/8\,x^2$$

Which has signature (1,1). This means that the origin is a saddle point.

3.7.1

$$f(x, y, z) = xyz, \quad F_1(x, y, z) = 2xy + 2xz + 2yz - 10 = 0, F_2(x, y, z) = y - 2x = 0$$

$$Df = [yz\,xz\,xy]\,,\, DF_1 = 2\,[y + z\; x + z\; x + y]\;\; DF_2 = [-2\,1\,0]$$

Solving for the equations:

$$Df = \lambda_1 DF_1 + \lambda_2 DF_2, F_1 = 0, F_2 = 0$$

we find

$$x = \frac{\sqrt{5}}{2\sqrt{2}}, \quad y = \frac{\sqrt{5}}{\sqrt{2}}, \quad z = \frac{\sqrt{5}}{\sqrt{2}}$$

3.7.2

$f(x, y, z) = x + y + z, F_1 = x - \sin(z)$

$Df = [1\,1\,1], DF_1 = [1\,0\; -\cos(z)]$

We cannot solve $Df = \lambda DF_1$. The vector $[0\,1\,0]$ is always tangent to the surface defined by $F_1 = 0$, so we can always increase f by going in this direction, or decrease it by travelling in the opposite direction.

3.7.3

$f(x, y, z) = x^3 + y^3 + z^3, \quad F_1(x, y, z) = x + y + z - 2, \quad F_2(x, y, z) = x + y - z - 3$

The equations

$Df = \lambda_1 DF_1 + \lambda_2 DF_2, F_1 = 0, F_2 = 0$

have the solutions:

$x = 2, y = \frac{1}{2}, z = -\frac{1}{2}$ and $x = 3, y = -\frac{1}{2}, z = -\frac{1}{2}$.

b. Restricting the function to the lines gives

$f(x) = \frac{15}{2}x^2 - \frac{75}{4}x + \frac{31}{2}\;\; f''(x) = 15 > 0$

So the critical point that we found is a restricted minimum.

3.7.6

$f(x,y,z) = 2\,xy + 2\,yz - 2\,x^2 - 2\,y^2 - 2\,z^2$

$F_1 = x^2 + y^2 + z^2 - 1$

The equations $Df = \lambda DF_1, F_1 = 0$ Have the solutions

$(x,y,z) = (1,0,0) \quad (x,y,z) = (-1,0,0) \quad (x,y,z) = \frac{1}{\sqrt{3}}(1,-1,1), \quad \frac{1}{\sqrt{3}}(1,1,1)$

3.7.7

$f(x,y,z) = xyz, \quad F_1(x,y,z) = ax + by + cz - 1$

$Df = \lambda DF_1$

has one solution

$z = \frac{ac}{a^2b+b^2c+ac^2}, x = \frac{ab}{a^2b+b^2c+c^2a}, y = \frac{bc}{a^2b+b^2c+ac^2}$

Any point on the x, y, or z axis will also be a critical point. The function $f(x,y,z) = 0$ on these axis, but in any neighborhood of a point on the axis we can find positive and negative values,