4.10.18 $z^2 = x^2 + y^2$ defines a cone which makes an angle of $\frac{\pi}{4}$ with the z axis. The region is spherical coordinates is

$$S = \{(r, \theta, \phi)|0 \leq r \leq 4, 0 \leq \phi \leq \frac{\pi}{4}, 0 \leq \theta \leq 2\pi\}$$

$$V = \int_S r^2 \sin\phi dr d\phi d\theta = 64\pi(2 - \sqrt{2})$$

4.10.19 $\Phi(u, v) = (u - v^2, u^2 + v)$ The boundaries of Q are $u = 0$, $u = 1$, $v = 0$, $v = 1$

$\Phi(0, v) = (-v^2, v)$ $\Phi(1, v) = (1 - v^2, 1 + v)$

$\Phi(u, 0) = (u, u^2)$ $\Phi(u, 1) = (u - 1, u^2 + 1)$

These 4 curves make kind of a rectangle with parabolas for sides.

b. Show Φ is one to one.

Suppose $\Phi(u_1, v_1) = \Phi(u_2, v_2)$

$u_1 - u_2 - v_1^2 + v_2^2 = u_1^2 - u_2^2 + v_1 - v_2 = 0$

If $u_1 = u_2$ then $v_1 = v_2$. Without loss of generality we can assume $u_1 > u_2$. Then $v_1 > v_2$ and $v_1 < v_2$ which is a contradiction.

c. $\int_A x|dxdy| = \int_Q (u - v^2)|detD(\Phi)||dudv|$

$= \int_Q (u - v^2)(1 + 4uv)dudv = \frac{1}{3}$

3.1.1 $F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ $[D(F)] = [2x\, 2y\, 2z]$

$[D(F)]$ fails to be onto only if $x = y = z = 0$ and $F(0, 0, 0) = -1 \neq 0$ so this point is no on our manifold.

3.1.2 $F(x, y) = x + x^2 + y^2 - 2$

$[D(F)] = [2x + 1\, 2y]$

This fails to be onto is $x = -\frac{1}{2}, y = 0$ but $F(-\frac{1}{2}, 0) = -\frac{5}{4} \neq 0$ so this point is not on the surface.

3.1.5

$X_c = \{(x, y)|x^2 + y^3 = c\}$

$[D(F)] = [2x\, 3y^2]$

$[D(F)]$ fails to be onto if $x = 0, y = 0$ which is in X_c only if $c = 0$

3.1.8 $X_a = \{(x, y, z)|x - y^2 = a\}$ $Y_b = \{(x, y, z)|x^2 + y^2 + z^2 = b\}$

Let $F_a = x - y^2 - a$, $G_b = x^2 + y^2 + z^2 - b$

$[D(F_a)] = [1\, -2y\, 0]$

$[D(G_b)] = [2x\, 2y\, 2z]$

$[D(F_a)]$ always has a nonzero entry so it is always onto and defines a smooth manifold.

$[D(G_b)]$ is onto whenever $b \neq 0$

b. $[D\Phi] = \begin{bmatrix} 1 & -2y & 0 \\ 2x & 2y & 2z \end{bmatrix}$

$[D\Phi]$ fails to be onto if $[1\, -2y\, 0]$ and $[2x\, 2y\, 2z]$ are lineaerly dependent. This happens is $z = 0, 2x = -1, x = \frac{-1}{2}$ or $z = 0$ and $y = 0$. This happens if $a^2 = b$ or if $a + b = \frac{-1}{4}$

3.1.14 For what value of c is $\sin(x+y) = c$ a smooth curve.

First off is $|c| > 1$ then $\sin(x+y) = c$ has no solutions. Otherwise the locus of this equation is an infinite number of lines $x + y = \sin^{-1}(c) + 2n\pi, n \in \mathbf{N}$, and $x + y = \pi - \sin^{-1}(c) + 2n\pi n \in \mathbf{N}$

5.1.1 What is vol_3 of the 3 parallelogram in $\mathbf{R}^4$ spanned by $(1,0,1,1), (0,2,1,1), (1,1,0,2)$?.
$\sqrt{det(V^tV)} = \sqrt{30}$

5.1.2 This volume is $\sqrt{9\sqrt{6} - 18}$

5.1.3 Suppose $v_k = \sum_{i=1}^{k-1} a_i v_i$. The last column of V^TV is

$$\begin{bmatrix} v_1 \circ v_k \\ \ldots \\ v_k \circ v_k \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{k-1} a_i v_1 \circ v_i \\ \ldots \\ \sum_{i=1}^{k-1} a_i v_k \circ v_i \end{bmatrix} = \sum_{i=1}^{k-1} a_i \begin{bmatrix} v_1 \circ v_i \\ \ldots \\ v_k \circ v_i \end{bmatrix}$$

This last column is a linear combination of the previous columns, so $Det(V^TV) = 0$

Another way to prove this is using the rank. Since v_i are linearly dependent we have that $Rank(V) < k$ and also that $Rank(V^TV) < k$. Since V^TV is a $k \times k$ matrix with less than full rank $det(V^TV) = 0$