

5.2.4 a. $F((x, y, z)) = x^2 + y^2 + z^2 - 2R\sqrt{x^2 + y^2} + R^2 - r^2 = 0$

$$DF = [2x - \frac{2Rx}{\sqrt{x^2+y^2}}, 2y - \frac{2Ry}{\sqrt{x^2+y^2}}, 2z]$$

This fails to be onto if $z = 0, 2x = \frac{2Rx}{\sqrt{x^2+y^2}}, 2y = \frac{2Ry}{\sqrt{x^2+y^2}}$

If $z = 0$ then $x^2 + y^2 = (R - r)^2$ or $x^2 + y^2 = (R + r)^2$, and in particular $\frac{R}{\sqrt{x^2+y^2}} \neq 1$. The only way for DF to now fail to be onto is if $(x, y, z) = (0, 0, 0)$ which is not on the torus $r < R$

b. An embedding is

$$\eta(\theta, \phi) = (R \cos(\theta) + r \cos(\theta) \cos(\phi), R \sin(\theta) + r \sin(\theta) \cos(\phi), r \sin(\phi)), \theta, \phi \in [0, 2\pi)$$

c.

$$z > 0, \sin(\phi) > 0, 0 < \phi < \pi$$

$$x > 0, y > 0, 0 < \theta < \frac{\pi}{2}$$

$$z > x + y, r \sin(\phi) > R \cos(\theta) + r \cos(\theta) \cos(\phi) + R \sin(\theta) + r \sin(\theta) \cos(\phi)$$

5.3.1 $\gamma(t) = (r(t), \theta(t))$ In polar coordinates.

$$\gamma(t) = (r(t) \cos(\theta(t)), r(t) \sin(\theta(t)))$$

$$D\gamma = (r' \cos(\theta) - r\theta' \sin(\theta), r' \sin(\theta) + r\theta' \cos(\theta))$$

$$Det([D\gamma]^T [D\gamma]) = r'^2 + r^2 \theta'^2$$

$$l = \int_a^b \sqrt{Det([D\gamma]^T [D\gamma])} dt = \int_a^b \sqrt{r'^2 + r^2 \theta'^2} dt$$

b. $L = \int_0^a \sqrt{\alpha^2 e^{-2\alpha t} + e^{-2\alpha t}} dt = \frac{1-e^{-\alpha a}}{\alpha} \sqrt{\alpha^2 + 1}$

c. We can see that $\lim_{a \rightarrow \infty} L(a) = \frac{\sqrt{\alpha^2 + 1}}{\alpha} < \infty$. The tangent sweeps through a whole circle everytime a increases by 2π so this continues to spiral in towards the origin, but the entire curve had a bounded arc length.

5.3.3 $\gamma(t) = (r \sin(\phi) \cos(\theta), r \sin(\phi) \sin(\theta), r \cos(\phi))$

$$L = \int_a^b \sqrt{r'^2 + r^2 \phi'^2 + r^2 \theta'^2 \sin^2(\phi)} dt$$

b. $\gamma(t) = (\cos(t), \tan(t), \frac{\pi}{2} - t)$

The integrand is $\sqrt{\sin^2(t) + \cos^2(t) + \cos^2(t) \sec^4(t) \sin^2(\frac{\pi}{2} - t)} = \sqrt{2} L(a) = \int_0^a \sqrt{2} dt = a\sqrt{2}$

5.3.5

$$\gamma(x, y) = (x, y, \frac{x^2}{4} + \frac{y^2}{9}) \det([D\gamma]^T [D\gamma]) = 1 + \frac{x^2}{4} + \frac{4y^2}{81}$$

$$S = \int_A \sqrt{1 + \frac{x^2}{4} + \frac{4y^2}{81}} |dx dy| = \int_{-2a}^{2a} \int_{-3\sqrt{a^2-x^2/4}}^{3\sqrt{a^2-x^2/4}} \sqrt{1 + \frac{x^2}{4} + \frac{4y^2}{81}} dy dx$$

b. The region in (a). can be described as the image of the region $S = \{(x, y, z) | x^2 + y^2 \leq z \leq a^2\}$

under the transformation $T T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The volume of V is $Vol(V) = \det[T] vol(S) = 6 vol(S)$

$$Vol(s) = \int_0^{2\pi} \int_0^a \int_{r^2}^{a^2} r dz dr d\theta = \frac{\pi a^4}{2}$$

$$Vol(V) = 3\pi a^4$$

5.3.7

a. $\eta((\phi, \theta)) = (a \sin(\phi) \cos(\theta), b \sin(\phi) \cos(\theta), c \cos(\theta))$

b.

$$\int_0^\pi \int_0^{2\pi} \sqrt{a^2 b^2 \cos^2(\phi) \sin^2(\phi) + c^2 \sin^4(\phi)(b^2 \cos^2(\theta) + a^2 \sin^2(\theta))} d\theta d\phi$$

6.1.1 0-forms $\phi = 1$

1-forms dx_1, dx_2, dx_3

2-forms $dx_1 \wedge dx_2, dx_1 \wedge dx_3, dx_2 \wedge dx_3$

3-forms $dx_1 \wedge dx_2 \wedge dx_3$

b. Here I'll use some quick notation to get through the lists. $(i,j,\dots) = dx_i \wedge dx_j \dots$

0-form 1

1-forms (1) (2) (3) (4)

2-forms (1,2) (1,3) (1,4) (2,3) (2,4) (3,4)

3-forms (1,2,3) (1,2,4) (1,3,4) (2,3,4)

4-forms (1,2,3,4)

c. There are 5 elementary 4 forms on \mathbf{R}^5 (1,2,3,4) (1,2,3,5) (1,2,4,5) (1,3,4,5) (2,3,4,5)

2. $(dx_3 \wedge dx_2 \wedge dx_4)(v_1, v_2, v_3) = 4.$

3 $dy((1,2)) = 6$

6. a. $dx_1 \wedge dx_2(v_1, v_2) = 3$

b. Silliness. There is no dx_3

c. Silliness. A 2-form takes 2 vectors, not 3.

d. Silliness. No x_4

e. Silliness A 3-form takes 3 vectors.

f. $(dx_1 \wedge dx_2 \wedge dx_3)(v_1, v_2, v_3) = \det(v_1, v_2, v_3) = -35$

7. a. 0-forms 1

1-forms 4

2-forms 6

3-forms 4

4-forms 1

b. \mathbf{R}^5 1-forms 5

2-forms 10

3-forms 10

4-forms 5

5-forms 1

c. $\dim(A^3(E)) = \binom{7}{3} = 35$

d. $\dim(A^3(E)) = 1$

6.1.13

$$\varphi \wedge \psi(v_1, v_2, v_3, v_4) = \varphi(v_1, v_2)\psi(v_3, v_4) - \varphi(v_1, v_3)\psi(v_2, v_4) + \varphi(v_1, v_4)\psi(v_2, v_3) - \varphi(v_2, v_4)\psi(v_1, v_3) + \varphi(v_3, v_4)\psi(v_1, v_2) + \varphi(v_2, v_3)\psi(v_1, v_4)$$