# Homework #6 Solutions

Note: $\ominus$ = "convergent" & $\mathcal{D}$ = "divergent"

## PART A

1. By a previous HW problem we know that if $\sum a_n$ is a positive series which is convergent then $\sum \sin(a_n)$ is convergent. Indeed $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent positive series by the p-test $\therefore \sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ is convergent.

We could also use the Limit comparison theorem (LCT):

We have $\lim_{n\to\infty} \dfrac{\sin\left(\frac{1}{n^2}\right)}{1/n^2} \overset{\text{L'Hopital}}{=} \lim_{n\to\infty} \dfrac{-2\frac{1}{n^3}\cos\left(\frac{1}{n^2}\right)}{-2\frac{1}{n^3}} = \lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right)$

$\overset{\text{continuity of } \cos(x)}{=} \cos\left(\lim_{n\to\infty}\frac{1}{n^2}\right) = \cos(0) = 1 > 0$. By LCT we have

$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ is $\ominus \iff \sum_{n=1}^{\infty} \frac{1}{n^2}$ is $\ominus$. As noted above,

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is $\ominus$ by the p-test, so $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ is $\ominus$.

2a) Observe that for each $n \geq 1$, $\frac{1}{n} \leq 1$ & so $0 \leq \frac{1}{n}\cdot\frac{1}{2^n} \leq \frac{1}{2^n}$.

We know $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent by [insert favorite test/theorem].

(Really because it's a geometric series, but each of our tests can be used to show convergence)

Thus $\sum_{n=1}^{\infty} \frac{1}{n}\frac{1}{2^n}$ is $\ominus$ by the comparison theorem.

It follows that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\cdot\frac{1}{2^n}$ is $\ominus$ since it is absolutely $\ominus$.

2b) We can see that $a_n := \frac{1}{n 2^n}$ is strictly decreasing since for every $n \in \mathbb{N}$ we have

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)2^{(n+1)}}}{\frac{1}{n2^n}} = \frac{n \cancel{2^n}}{(n+1)2 \cdot \cancel{2^n}}$$

$$= \frac{1}{2} \cdot \frac{n}{(n+1)} < 1$$

& so $a_{n+1} < a_n$ for each $n \in \mathbb{N}$.

Moreover $\lim_{n \to \infty} \frac{1}{n2^n} = 0$. Thus we may apply the Alternating series test which tells us that for each $n \geq 2$:

$$|s_{n-1} - L| < a_n.$$

Thus we are guarenteed that $|s_{n-1} - L| < \frac{1}{10^3}$ if $a_n < \frac{1}{10^3}$. That is $\frac{1}{n2^n} < \frac{1}{10^3} \iff 10^3 < n2^n$.

Plugging in some values for $n$ we find

$$7 \cdot 2^7 = 896 < 1000 \quad \text{so } n_0 > 7,$$

but $8 \cdot 2^8 = 2048 > 1000$ as desired.

Thus $\underline{n_0 = 8}$.

3. The series is

$$1 - \frac{1\cdot 3}{3!} + \frac{1\cdot 3\cdot 5}{5!} - \frac{1\cdot 3\cdot 5\cdot 7}{7!} + \cdots + (-1)^n \frac{1\cdot 3\cdot 5 \cdots (2n+1)}{(2n+1)!} + \cdots$$

Now $\dfrac{1\cdot 3\cdot 5\cdots(2n+1)}{(2n+1)!} = \dfrac{1}{2\cdot 4\cdot 6\cdots(2n)} = \dfrac{1}{2n!}$.

The series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2n!} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{2}e^1$ so it is absolutely convergent & thus $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{1}{2n!} = \sum_{n=0}^{\infty}(-1)^n \frac{1\cdot 3\cdots(2n+1)}{(2n+1)!}$ is

convergent.

4. We have $a_1 = 1$ & $a_{n+1} = \left(\sum_{j=1}^{n} 3^{-j}\right)a_n$.

<u>Claim</u>: $\sum_{n=1}^{\infty} a_n$ is convergent

<u>Pf of claim</u>: Note: since $a_1 > 0$ & $\sum_{j=1}^{n} 3^{-j} > 0$ $\forall n$,
we see that $a_n > 0$ $\forall n$.
We have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\left(\sum_{j=1}^{n}3^{-j}\right)\cancel{a_n}}{\cancel{a_n}} = \lim_{n\to\infty}\sum_{j=1}^{n}3^{-j} = \ldots$$

$$\ldots = \sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2} < 1.$$

By the ratio test $\sum a_n$ is <u>convergent</u>.

PART B

5. a) Pf: We are given that for each $n$, $a_n > 0$, $b_n > 0$ & that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ & $\sum b_n$ is convergent.

Since $a_n, b_n$ are positive $\forall n$, $\frac{a_n}{b_n} > 0$ $\forall n$.

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \Rightarrow$ (by definition of limit) that

$\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$ $\frac{a_n}{b_n} = \left|\frac{a_n}{b_n} - 0\right| < 1$.

Thus $\forall n \geq n_0$ $0 < a_n < b_n$.

By the comparison theorem we have $\sum_{n=n_0+1}^{\infty} a_n$ is convergent since $\sum_{n=n_0+1}^{\infty} b_n$ is convergent.

Of course, $\sum_{n=1}^{n_0} a_n$ is a finite sum so it is finite.

Finally we see that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{n_0} a_n + \sum_{n=n_0+1}^{\infty} a_n$ is convergent.

b) The converse of a) is "If $\sum a_n$ is convergent Then $\sum b_n$ is convergent." Where we still have the assumption that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

This is false. Here is a counter-example:
let $a_n = 0$ $\forall n$ & let $b_n = 1$ $\forall n$.
Then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ & $\sum a_n$ is convergent, but $\sum b_n$ is D.

#6. a) Let $a_n = \dfrac{1}{n^{1.01} - n^{0.99}}$ & let $b_n = \dfrac{1}{n^{1.01}}$.

Then $\sum_{n=2}^{\infty} b_n$ is $\ominus$ by the p-test.

We have

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{1/n^{1.01}}{1/(n^{1.01} - n^{.99})} = \lim_{n\to\infty} \frac{n^{1.01} - n^{.99}}{n^{1.01}}$$

$$= \lim_{n\to\infty} 1 - n^{-.02} = 1 > 0.$$

By the limit comparison theorem $\sum_{n=2}^{\infty} a_n$ is $\ominus$ $\iff$ $\sum_{n=2}^{\infty} b_n$ is $\ominus$. We know $\sum_{n=2}^{\infty} b_n$ is $\ominus$,

$\therefore$ $\sum_{n=2}^{\infty} a_n$ is $\ominus$.

b) We'll fix things so that $a_2 = 1$, & $a_{n+1} = \dfrac{(\ln(n))^2}{n} a_n$ for every $n \geq 2$. Let's try the ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(\ln(n))^2}{n}\cancel{a_n}}{\cancel{a_n}} = \lim_{n\to\infty} \frac{(\ln(n))^2}{n}$$

$$\overset{\text{L'Hopital}}{=} \lim_{n\to\infty} \frac{\frac{2}{n}\ln(n)}{1} = 2\lim_{n\to\infty} \frac{\ln(n)}{n} \overset{\text{L'Hopital}}{=} 2\lim_{n\to\infty} \frac{1}{n} = 0 < 1.$$

By the ratio test the series is convergent.

#6 c) Observe

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{(n+1)^{n+1}}{(2(n+1))!}}{\dfrac{n^n}{(2n)!}}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \frac{(n+1)}{(2n+2)(2n+1)}$$

(since $\left(\frac{n+1}{n}\right)^n$ & $\frac{1}{2n+1}$ are convergent sequences)

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \frac{1}{2(2n+1)} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \lim_{n\to\infty} \frac{1}{2(2n+1)}$$

$$= e \cdot 0 = 0 < 1$$

By the ratio test $\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$ is C.

#7 a) As we have seen, $\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{1/n} = 1$ (By L'Hopital).

Since $f(x) = x^2$ is continuous

$$\lim_{n\to\infty} \frac{(\sin(\frac{1}{n}))^2}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{\sin(\frac{1}{n})}{\frac{1}{n}}\right)^2$$

$$= \left(\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}}\right)^2$$

$$= 1^2 = 1 > 0.$$

By the Limit comparison theorem we see that $\sum_{n=1}^{\infty} \sin(\frac{1}{n})^2$ is convergent since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is $\ominus$. Thus $\sum_{n=1}^{\infty} (-1)^{n+1} (\sin(\frac{1}{n}))^2$ is Absolutely $\ominus$.

b) This series is not absolutely convergent by the p-test. However the sequence $\frac{1}{n^{4/5}}$ is monotonically decreasing to $0$. Since for every $n$, $(-1)^{n^2} = (-1)^n$ we see that the series $\sum_{n=1}^{\infty} (-1)^{n^2} n^{-4/5}$ is alternating. By the alternating series test we have that $\sum^{\infty} (-1)^{n^2} n^{-4/5}$ is convergent.

7 c) First,

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n}{n^2+1}} = \lim_{n\to\infty} \frac{n^2+1}{n^2} = \lim_{n\to\infty} 1 + \frac{1}{n^2} = 1 > 0.$$

By LCT $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is $\mathcal{D}$ since $\sum_{n=1}^{\infty} \frac{1}{n}$ is $\mathcal{D}$. Thus $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ is <u>not</u> absolutely $\mathcal{C}$.

We have $\frac{n}{n^2+1} = \frac{\frac{1}{n}}{\frac{1}{n}} \left( \frac{n}{n^2+1} \right) = \frac{1}{n + \frac{1}{n}}$. Well

$$n+1 + \frac{1}{n+1} > n + \frac{1}{n} \quad \text{since} \quad 1 + \frac{1}{n+1} > \frac{1}{n}.$$

Thus

$$\frac{1}{(n+1) + \frac{1}{(n+1)}} < \frac{1}{n + \frac{1}{n}} \quad \& \text{ so } \quad \frac{n}{n^2+1}$$

is strictly decreasing. It is clear that

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}} = 0.$$

Thus by the AST, the series is conditionally $\mathcal{C}$.