COUNTING SETS: LEVELS OF INFINITY

RECALL $A \neq \phi$ IS FINITE IF $\exists n \in \mathbb{N}$ ST. A HAS PRECISELY n ELEMENTS

$A = \{a_i \mid 1 \leq i \leq n\}$ & $i \neq j \Rightarrow a_i \neq a_j$

ANOTHER WAY: IF $\mathbb{N}_n = \{1, 2, \ldots, n\}$ THEN $f : \mathbb{N}_n \to A$, $f(i) = a_i$, IS A 1-TO-1 AND ONTO FUNCTION.

SUCH FUNCTIONS CALL BIJECTIONS

DEFINITION $f : C \to D$ IS 1-TO-1 IF $\forall c_1, c_2 \in C.\ c_1 \neq c_2 \Rightarrow f(c_1) \neq f(c_2)$

$f : C \to D$ IS ONTO IF $\forall d \in D.\ \exists c \in C$ ST $f(c) = d$.

THIS IS THE CORRESPONDENCE DEFINITION OF FINITE: $A \neq \phi$ IS FINITE IF $\exists n \in \mathbb{N}$ AND A BIJECTION $f : \mathbb{N}_n \to A$.

EXAMPLES

① $g_j, f_j : \mathbb{N} \to \mathbb{N}$ $f_j(n) = n + j$ OR $g_j(n) = jn$

ARE BOTH 1-TO-1

WHERE j FIXED $\in \mathbb{N}$,

BUT NOT ONTO. IF $j \neq 1$. SUPPOSE

$f_j(n_1) = f_j(n_2)$ $\therefore$ $n_1 + j = n_2 + j$ & SO $n_1 = n_2$

THIS IS CONTRAPOSITIVE OF $n_1 \neq n_2$

$\Rightarrow f_j(n_1) \neq f_j(n_2)$. SINCE $jn_1 = jn_2$

$\Rightarrow n_1 = n_2$, g_j IS ALSO 1-TO-1.

IF $j \neq 1$, $f_j(\mathbb{N}) = \{n + j \mid n \in \mathbb{N}\} = \{1+j, 2+j,$

$3+j, \ldots\}$, AND $1, 2, \ldots, j \notin f_j(\mathbb{N})$. NOTE

TRUE EVEN IF $j = 1$.

IN THIS CASE,

$g_j(\mathbb{N}) = \{jn \mid n \in \mathbb{N}\} = \{j, 2j, 3j, \ldots\}$

AND $1, 2, \ldots, j-1 \notin g_j(\mathbb{N})$. IF $j = 1$, $g_j(n)$

$= 1 \cdot n = n$, AND g_1 IS ONTO ($\therefore$ A BIJECTION)

② SUPPOSE A FINITE & $f : \mathbb{N}_n \to A$

A BIJECTION. THEN $g : \mathbb{N}_n \to A$

$g(j) = f(a_{n+1-j})$ IS ANOTHER BIJECTION

SO THERE ARE MANY POSSIBLE BIJECTIONS

WE CAN USE THESE IDEAS TO REFINE OUR NOTION OF INFINITE (PREVIOUS DEFN: NOT FINITE)

DEFINITION A IS **COUNTABLY INFINITE** IF $\exists f : \mathbb{N} \longrightarrow A$ ST f IS A BIJECTION. SETTING $f(i) = a_i$, WE CAN EXPRESS $A = \{a_i \mid i \in \mathbb{N}\}$ AND $i \neq j \Rightarrow a_i \neq a_j$

EXAMPLES ① THE ODDS O & EVENS E POSITIVE INTEGERS ARE COUNTABLY INFINITE.

$f : \mathbb{N} \longrightarrow E$, $f(n) = 2n$ $g : \mathbb{N} \to O$, $g(n) = 2n-1$ ARE BIJECTIONS (EXERCISE)

② THE INTEGERS $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ ARE COUNTABLY INFINITE. TRICK E AND O ARE DISJOINT COUNTABLY INFINITE SUBSETS OF $\mathbb{N}$. DEFINE $f(2n) = n-1$ AND $f(2n-1) = -n$, FOR $n \in \mathbb{N}$

So $f(E) = \{0, 1, 2, 3, \ldots\}$: $f(2)=$

$f(4) = 1$, $f(6) = 2$, $f(8) = 3$, ..., AND

$f(O) = \{-1, -2, -3, \ldots\}$: $f(1) = f(2-1) = -$

$f(3) = f(2\cdot 2 - 1) = -2, \ldots$ THUS,

$f: \mathbb{N} \longrightarrow \mathbb{Z}$ IS A BIJECTION

A MORE SUPRIZING RESULT IS:

<u>THM</u> THE POSITIVE RATIONAL NUMBERS $\mathbb{R}_1 = \left\{ \frac{p}{q} \;\middle|\; \begin{array}{l} 0 \le p \le q, \\ p, q \text{ IN LOWEST TERMS} \end{array} \right\}$ IS COUNTABLY INFINITE

<u>PF</u> SINCE WE WANT $\frac{p}{q}$ IN LOWEST TERMS, WE ENUMERATE ACCORDING TO INCREASING DENOMINATORS

$$\overset{1}{1}, \overset{2}{\frac{1}{2}}, \overset{3}{\frac{1}{3}}, \overset{4}{\frac{2}{3}}, \overset{5}{\frac{1}{4}}, \cancel{\frac{2}{4}}, \overset{6}{\frac{3}{4}}, \overset{7}{\frac{1}{5}}, \overset{8}{\frac{2}{5}}, \overset{9}{\frac{3}{5}}, \overset{10}{\frac{4}{5}},$$

$$\overset{11}{1/6}, \cancel{2/6}, \cancel{3/6}, \cancel{4/6}, \overset{12}{\frac{5}{6}}, \ldots$$

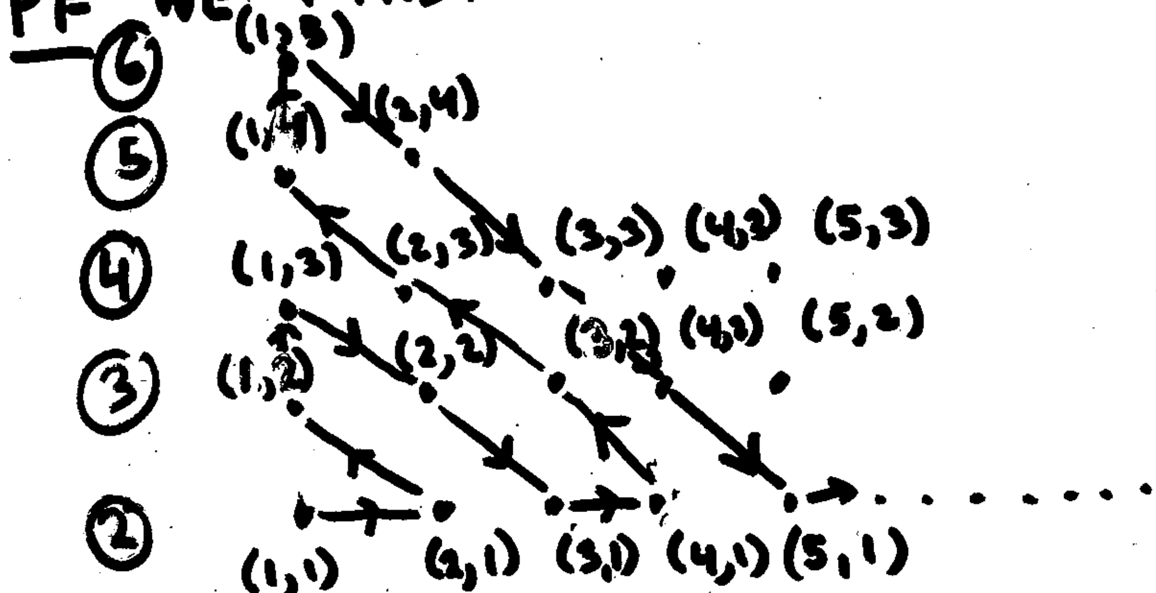
ALTHOU WE CANT EASILY FIND A FORMULA FOR $f: \mathbb{N} \to \mathbb{R}$, THE ABOVE PATTERN SHOWS HOW TO FIND $f(n)$ FOR ANY n:

$f(5) = 1/4$, $f(11) = \frac{1}{2}$, $f(15) = \frac{3}{\ }$[illegible], $f(19) = \frac{1}{8}$

CONSIDER THE PRODUCT $\mathbb{N} \times \mathbb{N}$ $= \{(a,b) \mid a, b \in \mathbb{N}\}$: ALL POINTS IN THE PLANE. AN EVEN MORE SURPRIZING RESULT IS

<u>THM</u> $\mathbb{N} \times \mathbb{N}$ IS COUNTABLY INFINITE

<u>PF</u> WE FIRST VISUALIZE $\mathbb{N} \times \mathbb{N}$



WE NOW LIST FOLLOWING "DIAGONALS" THE $1^{\underline{ST}}$ DIAGONAL HAS 1 DOT, $2^{\underline{ND}}$ 2 DOTS, $3^{\underline{RD}}$ 3 DOTS, ... n^{TH} HAS n DOTS. SO AFTER COVERING n DIAGONALS WE WILL HAVE USED $1+2+\cdots+n = \frac{n(n+1)}{2}$ POSITIVE INTEGERS. THIS DEFINES A BIJECTION $g: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. THERE IS A FORMULA (HARD! EXERCISE)

EXAMPLE SHOW THAT $A = \{ \frac{p}{q} \in \mathbb{R} \mid 1 \le p \le 2 \} \subset \mathbb{R}$, COUNTABLY INFINITE
$p \le q$

SETTING $p=1$ $B = \{ \frac{1}{n} \mid n \in \mathbb{N} \} \subset A$

SETTING $p=2$, $\frac{2}{2n} = \frac{1}{n}$ ALREADY $\in B$

SO $C = \{ \frac{2}{2n+1} \mid n \in \mathbb{N} \}$ AND

$A = B \cup C$, $B \cap C = \phi$. DEFINE

$f : \mathbb{N} \longrightarrow A = B \cup C$ BY $f(2n-1) = \frac{1}{n}$

$f(2n) = \frac{2}{2n+1}$. So $f(O) = B$ & $f(E) = C$

CHECK THAT f IS A BIJECTION.

WE KNOW THAT THERE ARE MANY COUNTABLY INF. SUBSETS OF A COUNT INF SET B: IF $B = \mathbb{N}$, E, O, $\{ n \mid n \ge k \}$, ETC. WE ACTUALLY HAVE THE FOLLOWING

THM IF A COUNTABLY INFINITE AND $B \subset A$ IS NOT FINITE, THEN B IS COUNTABLY INFINITE

PRF LET $A = \{a_\lambda \mid \lambda \in \mathbb{N}\}$, LOOK AT $C_1 = \{\lambda \mid a_\lambda \in B\} \subseteq \mathbb{N}$. $C_1 \neq \phi \Rightarrow$ C_1 HAS SMALLEST VALUE λ_1. $C_2 = C_1 - \{\lambda_1\}$ $\neq \phi$ (SINCE B NOT FINITE) $\Rightarrow C_2$ HAS SMALLEST VALUE λ_2. CONTINUE:

SUPPOSE HAVE DEFINED $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ THEN $C_{n+1} = C_n - \{\lambda_1, \lambda_2, \dots \lambda_n\} \neq \phi$ AND LET $\lambda_{n+1} = \inf C_{n+1}$. THUS,

$C_1 = \{\lambda_n \mid n \in \mathbb{N}\}$ AND, $n_1 \neq n_2 \Rightarrow \lambda_{n_1} \neq \lambda_{n_2}$

SO $\mathbb{N} \xrightarrow{f} C_1$ IS A BIJECTION,
$n \longrightarrow \lambda_n$

AND SO IS $f : \mathbb{N} \longrightarrow B$.
$n \longrightarrow a_{\lambda_n}$

WE HAVE SEEN THAT $\mathbb{N} \times \mathbb{N}$ IS COUNTABLY INFINITE. THIS GIVES:

THM A_1, A_2 COUNTABLY INFINITE $\Longrightarrow A_1 \times A_2$ COUNTABLY INFINITE

THE INFINITY HOTEL PROF. GAUSS,

BESIDES DISCOVERING MATHEMATICS,
HAS A HOTEL. SINCE HE IS SO INTO
MATH, THIS HOTEL HAS A COUNTABLY
INFINITE # OF ROOMS. PROF G, BEING
SOMEWHAT ABSENT-MINDED, NEVER
REMEMBERS WHOSE IN WHAT ROOM.

1. A BUS PULLS UP WITH 5
PASSENGERS. HOW SHOULD PG
ASSIGN ROOMS SO THAT EVERYONE
HAS A SINGLE?

EXCEPT HAVE K
2 SAME AS 1,
PASSENGERS?

3 A BUS WITH AN INFINITE # OF
PASSENGERS (COUNTABLY INF)

4 2 BUSES, EACH WITH INFINITE #

NONE OF
5 GAUSS CLAIMS THAT
THESE WORRY HIM. IN FACT, HE
CLAIMS THAT HIS HOTEL CAN HANDLE
A COUNTABLY INF # OF BUSES, EACH
WITH A COUNT INF # OF PASSENGERS
IS GAUSS CORRECT, OR HAS HE BLOWN A
GASKET

$$
\begin{array}{lccccc}
B_1 & a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots \\
B_2 & a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots \\
B_3 & a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots \\
B_4 & a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

KNOW: $\exists$ BIJECTION

$$f : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$$

$$g : \mathbb{N} \longrightarrow \text{GUESTS}$$

$$g(n) = a_{f(n)}$$

(4/19) Recall that a set A is <u>countably infinite</u> if $\exists$ a bijection (i.e. a one-to-one and onto function) [NB:

$f : \mathbb{N} \to A$. Setting $f(i) = a_i$, we can express $A = \{a_i \mid i \in \mathbb{N}\}$, where $i \neq j \Rightarrow a_i \neq a_j$.

<u>Examples</u>

- The even natural numbers $\mathcal{E} = \{2, 4, 6, \dots\}$ and odd natural numbers $\mathcal{O} = \{1, 3, 5, \dots\}$ are countably infinite sets. Here are bijections:

$$f : \mathbb{N} \to \mathcal{E} \qquad\qquad f : \mathbb{N} \to \mathcal{O}$$
$$f(n) = 2n \qquad\qquad f(n) = 2n - 1$$

- The integers, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, form a countably infinite set. <u>Trick</u> In defining a bijection $f : \mathbb{N} \to \mathbb{Z}$, send $\mathcal{E}$ to the positive integers and $\mathcal{O}$ to the negative ones.

For $n \in \mathbb{N}$, define $\begin{cases} f(2n) = n-1 \\ f(2n-1) = -n. \end{cases}$ $\quad f(\mathcal{E}) = \{0, 1, 2, 3, \dots\}$, $f(\mathcal{O}) = \{-1, -2, -3, \dots\}$ and f is a bijection $\mathbb{N} \to \mathbb{Z}$.

- Define $R_1 = \{\frac{p}{q} \mid 0 \le p \le q,\ p \,\&\, q \text{ are integers}, \frac{p}{q} \text{ is in "lowest terms"}\}$

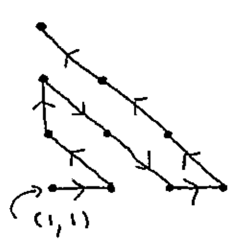
$=$ rational numbers between 0 and 1. R_1 is countably infinite. We enumerate R_1 by <u>increasing</u> <u>denominators</u>:

$n\ (\in \mathbb{N})$	1	2	3	4	5		6	7	8	...
$f(n)\ (\in R_1)$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\underbrace{\frac{2}{4}}_{=1/2}$	$\frac{3}{4}$	$\frac{1}{5}$	$\frac{2}{5}$	...

$1/2$, already counted: omit

We can't easily find a formula $f : \mathbb{N} \to R_1$, but the above pattern shows how to find $f(n)\ \forall n \in \mathbb{N}$. E.g. $f(5) = \frac{1}{4}$, $f(11) = \frac{1}{6}$, $f(15) = \frac{3}{7}$, $f(19) = \frac{1}{8}$.

- Consider the product $\mathbb{N} \times \mathbb{N} = \{(a,b) \mid a, b \in \mathbb{N}\}$, the set of natural-number lattice points in the plane.



$\mathbb{N} \times \mathbb{N}$ is countably infinite, and we define a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by zig-zagging up and down as shown (with arrows) in the picture at left.

<u>Hard exercise</u> Find a <u>formula</u> for f.

$f(1) = (1,1)$, $f(2) = (2,1)$, $f(3) = (1,2)$, $f(4) = (1,3)$, $f(5) = (2,2)$, ...

<u>Hilbert's Hotel</u>: Consider a hotel with a countably infinite number of rooms. The manager does not know which rooms are presently occupied.

- If five new guests arrive at once, the manager can accommodate them by moving every guest currently in room N to room $N+5$, opening up rooms $1, 2, 3, 4$, and 5, without double-booking.
- If a countably infinite number of guests arrive at once, move every guest currently in room N to room $2N$, then (if we enumerate the new guests $1, 2, \dots, \ell, \dots$) put new guest ℓ in room $2\ell - 1$ for all $\ell \in \mathbb{N}$.
- What happens if <u>2</u> countably infinite groups show up at once?

(4/21) One final comment about Hilbert's Hotel: since we know $\mathbb{N} \times \mathbb{N}$ is countably infinite, we can accommodate a countably infinite number of buses, each bearing a countably infinite number of guests!

<u>Recall</u> $R_1 = \mathbb{Q} \cap [0,1]$ is countably infinite.

Let $A = \{ \frac{p}{q} \in R_1 \mid p = 1 \text{ or } 2 \}$. We can write $A = B \cup C$, where $B = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$ and $C = \{ \frac{2}{2n+1} \mid n \in \mathbb{N} \}$; here $B \cap C = \emptyset$. Define $f : \mathbb{N} \to A = B \cup C$ by $f(2n-1) = \frac{1}{n}$, $f(2n) = 2/(2n+1)$; then $f(\mathcal{E}) = C$, $f(\mathcal{O}) = B$, and $f : \mathbb{N} \to A$ is a bijection, proving that A is countably infinite.

This is a special case of the following:

<u>Theorem.</u> If A is countably infinite, and $B \subset A$ is <u>not</u> finite, then B is countably infinite.

Sketch of <u>proof</u>: By hypothesis, A can be written $\{ a_i \mid i \in \mathbb{N} \}$. Consider the set $C_1 = \{ i \mid a_i \in B \} \subset \mathbb{N}$. Since $C_1 \neq \emptyset$, C_1 has a smallest element i_1. Since B is not finite, $C_2 = C_1 \setminus \{ i_1 \} \neq \emptyset$, so it has a smallest element i_2. Repeat this process: we see that $C_1 = \{ i_n \mid n \in \mathbb{N} \}$ is countably infinite, and that $f : \mathbb{N} \to B$, $f(n) = a_{i_n}$, is a bijection. ∎

Finally, we note that since $\mathbb{N} \times \mathbb{N}$ is countably infinite, we have:

<u>Theorem.</u> If A_1 and A_2 are countably infinite sets, so is $A_1 \times A_2$.

(Exam review 4/21 ; exam 3 on 4/23)