PROFESSIONAL PROBLEM 1 SOLUTIONS

1 a) <u>Pf</u>: Let $B = \{n \in \mathbb{N} \mid n \geq k\}$. We want to show that $A = B$.
By assumption, $A \subset \mathbb{N}$ & k is the smallest element of A.
It follows that $A \subset B$. We'll prove that $A = B$ by contradiction. Suppose $A \neq B$. Then the set $B \setminus A \neq \emptyset$.
(By definition $B \setminus A = \{x \in B \mid x \notin A\}$).
We are guaranteed that $B \setminus A$ has a smallest element by the well-orderedness of $\mathbb{N}$. Let p be the smallest element of $B \setminus A$, so $p \leq q \ \forall q \in B \setminus A$.
Now $k \in B \cap A \Rightarrow p > k$, & so $p-1 \geq k$. Since $k \leq p-1 < p$ we see that $p-1 \in B$, but $p-1 \notin B \setminus A$, thus $p-1 \in A$.
Finally assumption 2 tells us that $n \in A \Rightarrow n+1 \in A$.
Since $p-1 \in A$, it follows that $(p-1)+1 = p \in A$. But $p \in B \setminus A \Rightarrow p \notin A$. This is a contradiction. $\therefore$ $B \setminus A = \emptyset$, i.e. $A = B$.

1 b). <u>Pf</u>: Let $A = \{n \in \mathbb{N} \mid n \geq k \text{ \& } P(n) \text{ is true}\}$. Notice that

$\forall n \geq k$, $n \in A \Leftrightarrow P(n)$ is true. Thus to prove $\forall n \geq k$, $P(n)$ is true, it suffices to show that $A = \{n \in \mathbb{N} \mid n \geq k\}$. By assumption 1, we have that $P(k)$ is true & so $k \in A$. Moreover since $\forall n \in A$, $n \geq k$ we see that k is the smallest element of A.
Assumption 2 tells us that if $P(n)$ is true then $P(n+1)$ is true. In terms of A this gives us that $n \in A \Rightarrow n+1 \in A$. In summary, $A \subset \mathbb{N}$ & 1. k is the smallest element of A & 2. $n \in A \Rightarrow n+1 \in A$. By problem 1a) we conclude $A = \{n \in \mathbb{N} \mid n \geq k\}$, as desired.

2a) Pf: Base Case: Let $n=1$. Then $1^3 = \left(\frac{1(1+1)}{2}\right)^2$.

Inductive step: Suppose for some $n \in \mathbb{N}$ we have

$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Then we have

$$
\begin{aligned}
1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n^2 + 4(n+1))}{4} \\
&= \frac{(n+1)^2(n^2+4n+4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2.
\end{aligned}
$$

Thus if the formula is true for n then it's true for $n+1$. By induction the formula is true $\forall\, n \in \mathbb{N}$.

2b) Let $g(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$.

Let's compute $g(n)$ for the first several n:

$n=1 \Rightarrow g(n) = \frac{1}{1\cdot 2} = \frac{1}{2}$

$n=2 \Rightarrow g(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{4}{6} = \frac{2}{3}$

$n=3 \Rightarrow g(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{3}{4}$.

It seems reasonable to guess that $g(n) = \frac{n}{n+1}$.

If this is the case then $g(n) = 1 - f(n) \Rightarrow f(n) = 1 - \frac{n}{n+1} = \frac{1}{n+1}$.

Let's prove this result with induction:

Base case: Let $n=1$. Then $\frac{1}{1\cdot 2} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{1+1}$.

Inductive step: Say for some $n \in \mathbb{N}$ that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

Then

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n(n+2)+1}{(n+1)(n+2)}$$

$$= \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)}$$

2b) cont.

Continuing the computation:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}$$

$$=\frac{(n+1)^2}{(n+1)(n+2)}$$

$$=\frac{n+1}{n+2}$$

$$=1-\frac{1}{n+2}.$$

thus $f(n+1)=\frac{1}{n+2}$, as desired.

2c) Ans: We have

$$2^3+4^3+6^3+\cdots+(2n)^3$$

$$=2^3\cdot 1^3+2^3\cdot 2^3+2^3\cdot 3^3+\cdots+2^3\cdot n^3$$

$$=8(1^3+2^3+3^3+\cdots+n^3)$$

$$=8\left(\frac{n(n+1)}{2}\right)^2 \qquad [\text{by } 2a)]$$

$$=2n^2(n+1)^2.$$