Solutions for Professional Problem 2

1.

a) Ans: False

Counter-example: For each $n \in \mathbb{N}$ let $a_n = 0$ & $b_n = n$.

Then $\langle a_n \rangle$ converges, as does $\langle a_n b_n \rangle (= \langle a_n \rangle)$.

However $\langle b_n \rangle$ is divergent.

b) Ans: True.

Proof: Let $M = \lim_{n\to\infty} a_n b_n$. We claim that $\lim_{n\to\infty} b_n = \frac{M}{L}$.

In particular, $\langle b_n \rangle$ is convergent. Since for every n $a_n \neq 0$ & $L \neq 0$, theorem 2.1 (d) tells us that

$\langle \frac{(a_n b_n)}{a_n} \rangle = \langle b_n \rangle$ is a convergent sequence with

$\lim_{n\to\infty} \frac{a_n b_n}{a_n} = \frac{\lim_{n\to\infty} a_n b_n}{\lim_{n\to\infty} a_n} = \frac{M}{L}$, as desired.

c) Ans: True

Proof: Since $\langle b_n \rangle$ is bounded we may let $R \in \mathbb{R}$ be s.t. $|b_n| < R$ for each $n \in \mathbb{N}$.

By theorem 2.1 (b), since $\langle a_n \rangle$ is convergent to 0, the sequence $\langle Ra_n \rangle$ is convergent to 0. Likewise, problem 1 of homework 3 tells us $Ra_n \to 0 \iff |Ra_n| \to 0$. It is clear $-|Ra_n| \to 0$, also. Finally for each n we have

$$-|Ra_n| \leq b_n a_n \leq |Ra_n|.$$

By the Pinching theorem, we conclude that $b_n a_n \to 0$.

1 continued

d) Ans: True.

Proof: Let $\varepsilon = \frac{1-L}{2}$. Then $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$,

given ε $\exists\, n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow |a_n - L| < \varepsilon$.

Thus $n \geq n_0 \Rightarrow -\varepsilon < a_n - L < \varepsilon$

$$\Rightarrow -\left(\frac{1-L}{2}\right) < a_n - L < \frac{1-L}{2}.$$

In particular, $a_n - L < \frac{1-L}{2} \Rightarrow a_n < \frac{1}{2} + \frac{L}{2}$.

Since $L < 1$, $a_n < \frac{1}{2} + \frac{L}{2} \Rightarrow a_n < \frac{1}{2} + \frac{1}{2} = 1$.

Putting all of this together we have shown that $\exists\, n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow a_n < 1$, as desired.

e) Ans: True.

Proof: We note that the functions $f(x) = \cos(x)$, $g(x) = x^2$, & $h(x) = x^3$, are all continuous over the real numbers. Thus given a convergent sequences $\langle a_n \rangle, \langle b_n \rangle$ we have that $\langle g(a_n) \rangle = \langle a_n^2 \rangle$ & $\langle h(b_n) \rangle = \langle b_n^3 \rangle$ are convergent by Theorem 3.1. By theorem 2.1 (c) we have that since $\langle a_n^2 \rangle$ & $\langle b_n^3 \rangle$ are convergent sequences it follows that $\langle a_n^2 b_n^3 \rangle$ is convergent sequence. Finally, since $f(x) = \cos(x)$ is a continuous function theorem 3.1 tells us that $\langle \cos(a_n^2 b_n^3) \rangle$ is a convergent sequence.

2a) <u>Pf</u>: Let $S = \{a_n \mid n \in \mathbb{N}\}$.

(Note: S is a set not a sequence. For example, suppose $a_n = 0$ for each n. Then S would be the set consisting only of 0, i.e. $S = \{0\}$).

Since $\langle a_n \rangle$ is bounded below, the set S is also bounded below & thus S has an infimum. Let $r = \inf(S)$. We will now prove that

$a_n \longrightarrow 0$.

Let $\varepsilon > 0$. By definition of the infimum of a set we have that $r + \varepsilon$ is not a lower bound for S. Thus $\exists\, n_0 \in \mathbb{N}$ s.t. $a_{n_0} < r + \varepsilon$.

Since $\langle a_n \rangle$ is a decreasing sequence we have that $a_m \leq a_{n_0}$ whenever $m \geq n_0$. Putting this together with the above we see $\forall\, m \geq n_0$ $a_m \leq a_{n_0} < r + \varepsilon$ & so $a_m - r < \varepsilon$. Since r is a lower bound for S, $a_n - r \geq 0$ $\forall\, n \in \mathbb{N}$. Thus $a_n - r = |a_n - r|$.

Finally, we have given $\varepsilon > 0$ $\exists\, n_0 \in \mathbb{N}$ s.t. $m \geq n_0 \Rightarrow |a_m - r| < \varepsilon$. By definition, we have $\lim_{n \to \infty} a_n = r$, as desired.

2 b) i) Pf: Note that the function $f(x)=e^x$ is increasing for all $x \in \mathbb{R}$. By definition this means

$$x<y \iff e^x<e^y.$$

Let $b_n = \frac{n+3}{n+1}$ for each $n \in \mathbb{N}$, so $a_n = e^{b_n}$.

Then $b_n = \frac{n+3}{n+1} = 1 + \frac{2}{n+1}$. We see that $\forall n \in \mathbb{N}$

$$b_n - b_{n+1} = 1 + \frac{2}{n} - \left(1 + \frac{2}{n+1}\right)$$

$$= \frac{2}{n} - \frac{2}{n+1} = \frac{2n+2-2n}{n(n+1)} = \frac{2}{n(n+1)} > 0.$$

Thus $\forall n \in \mathbb{N}$ $b_n - b_{n+1} > 0$, i.e. $b_n > b_{n+1}$.

Since $f(x)=e^x$ is increasing we have

$$b_n > b_{n+1} \Rightarrow e^{b_n} > e^{b_{n+1}}$$

Thus for each n, $a_n > a_{n+1}$. That is, $\langle a_n \rangle$ is a decreasing sequence.

That $\langle a_n \rangle$ is bounded below follows from the fact that $e^x > 0$ $\forall x \in \mathbb{R}$.

ii) Ans: Notice that $\langle b_n \rangle$ defined above is convergent to 1. Since $f(x)=e^x$ is a continuous function Theorem 3.1 tells us that $\langle e^{b_n} \rangle$ $(=\langle a_n \rangle)$ is a convergent sequence & $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{b_n} = e^{\lim_{n\to\infty}(b_n)} = e!$

Thus $\lim_{n\to\infty} a_n = e$.