7.2

(4) $x' = y$
$y' = -\sin x - y$ $\Big\}$ $J(x,y) = \begin{bmatrix} 0 & 1 \\ -\cos x & -1 \end{bmatrix}$

$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = A$

$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + \lambda + 1 = (\lambda + \frac{1}{2} \pm \frac{\sqrt{3}}{2} i)$

$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ stable spiral

(8) $x' = x - 3y + 2xy$
$y' = 4x - 6y - xy$

Equilibrium points: $\left.\begin{array}{l} x - 3y + 2xy = 0 \\ 4x - 6y - xy = 0 \end{array}\right\}$ $9x - 15y = 0 \Rightarrow 3x = 5y$

$x - 3(\frac{3}{5}x) + 2x(\frac{3}{5}x) = 0 \Rightarrow 6x^2 - 4x = 0 \Rightarrow x = 0, \frac{2}{3}$

$\Rightarrow (0,0), (\frac{2}{3}, \frac{2}{5})$ eq. pts.

$J(x,y) = \begin{bmatrix} 1+2y & -3+2x \\ 4-y & -6-x \end{bmatrix}$

$(0,0)$; $A_1 = \begin{bmatrix} 1 & -3 \\ 4 & -6 \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -3 \\ 4 & -6-\lambda \end{vmatrix}$
$= \lambda^2 + 5\lambda + 6 = (\lambda+2)(\lambda+3)$

$\lambda = -2, -3 \Rightarrow$ Stable node

$(\frac{2}{3}, \frac{2}{5})$; $A_2 = \begin{bmatrix} \frac{9}{5} & -\frac{5}{3} \\ \frac{18}{5} & -\frac{20}{3} \end{bmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} \frac{9}{5} - \lambda & -\frac{5}{3} \\ \frac{18}{5} & -\frac{20}{3} - \lambda \end{vmatrix}$

$= \lambda^2 + \frac{73}{15}\lambda - 6$
$= (\lambda + \frac{73}{30} \pm \frac{\sqrt{73^2 + 24 \cdot 15^2}}{30})$

$\lambda = -\frac{73}{30} \pm \frac{\sqrt{73^2 + 24 \cdot 15^2}}{30} \Rightarrow$ saddle point

8.1

(6) $y'' + y = -2t^2 - 4 = -2(t^2+2)$, $y(0) = -2$, $y'(0) = 1$

$\Rightarrow y_h = c_1 \cos t + c_2 \sin t$, $y_p = -2t^2$

$y = c_1 \cos t + c_2 \sin t - 2t^2$

$-2 = y(0) = c_1$

$1 = y'(0) = c_2$

$y = -2\cos t + \sin t - 2t^2$

(14) $L(x) = \begin{bmatrix} 2t+5 \\ -6t-17 \end{bmatrix} = 2\begin{bmatrix} 1+t \\ -1-3t \end{bmatrix} + 3\begin{bmatrix} 1 \\ -5 \end{bmatrix} = 2L(x_1) + 3L(x_2)$

So the particular solution is $x_p = 2x_1 + 3x_2 = \begin{bmatrix} 2t+3 \\ 8 \end{bmatrix}$

$\left(L(2x_1 + 3x_2) = 2L(x_1) + 3L(x_2) = \begin{bmatrix} 2t+5 \\ -6t-17 \end{bmatrix} \right)$

(18) $x' = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ e^t \end{bmatrix}$

Find x_h: Solve $x_h' = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} x_h = Ax_h$

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$

$\lambda = -1, 3$

$\lambda = -1 \Rightarrow \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 1 & 2 & 0 \end{array}\right] \Rightarrow v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\lambda = 3 \Rightarrow \left[\begin{array}{cc|c} -2 & 4 & 0 \\ 1 & -2 & 0 \end{array}\right] \Rightarrow v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

$P^{-1}x_h = \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{3t} \end{bmatrix} \Rightarrow x_h = P\begin{bmatrix} c_1 e^{-t} \\ c_2 e^{3t} \end{bmatrix} = \begin{bmatrix} 2c_1 e^{-t} + 2c_2 e^{3t} \\ -c_1 e^{-t} + c_2 e^{3t} \end{bmatrix}$

Find x_p: Set $x_p = \begin{bmatrix} ae^t \\ be^t \end{bmatrix}$.

$x_p' = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} x_p + \begin{bmatrix} 0 \\ e^t \end{bmatrix} \Rightarrow \begin{bmatrix} ae^t \\ be^t \end{bmatrix} = \begin{bmatrix} (a+4b)e^t \\ (a+b)e^t \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \end{bmatrix}$

$\Rightarrow \left.\begin{array}{l} a = a+4b \\ b = a+b+1 \end{array}\right\} \Rightarrow \begin{array}{l} b = 0 \\ a = -1 \end{array}$

$x_p = \begin{bmatrix} -e^t \\ 0 \end{bmatrix}$

$x = x_h + x_p = \begin{bmatrix} 2c_1 e^{-t} + 2c_2 e^{3t} - e^t \\ -c_1 e^{-t} + c_2 e^{3t} \end{bmatrix}$

8.2

⑥ $y'' - 2y' + 2y = e^t \sin t$

Find y_h ; $y'' - 2y' + 2y = 0 \Rightarrow r^2 - 2r + 2 = 0 \Rightarrow r = 1 \pm i$

$$y_h = c_1 e^t \cos t + c_2 e^t \sin t$$

$$y_1 = e^t \cos t \ , \ y_2 = e^t \sin t$$

$$v_1' = \begin{vmatrix} 0 & e^t \sin t \\ e^t \sin t & e^t \sin t + e^t \cos t \end{vmatrix} \Big/ \begin{vmatrix} e^t \cos t & e^t \sin t \\ e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t \end{vmatrix}$$

$= -e^{2t} \sin^2 t \ / \ e^{2t}$

$= -\sin^2 t$

$= \dfrac{\cos 2t - 1}{2}$

$\Rightarrow v_1 = \int \dfrac{\cos 2t - 1}{2}\, dt = \dfrac{\sin 2t}{4} - \dfrac{t}{2}$

$$v_2' = \begin{vmatrix} e^t \cos t & 0 \\ e^t \cos t - e^t \sin t & e^t \sin t \end{vmatrix} \Big/ \begin{vmatrix} e^t \cos t & e^t \sin t \\ e^t \cos t - e^t \sin t & e^t \sin t + e^t \cos t \end{vmatrix}$$

$= \sin t \cos t$

$= \dfrac{\sin 2t}{2}$

$\Rightarrow v_2 = \int \dfrac{\sin 2t}{2}\, dt = -\dfrac{\cos 2t}{4}$

$y_p = v_1 y_1 + v_2 y_2 = \left(\dfrac{\sin 2t}{4} - \dfrac{t}{2}\right)(e^t \cos t) + \left(-\dfrac{\cos 2t}{4}\right)(e^t \sin t)$

$= -\frac{1}{2} t e^t \cos t + \frac{1}{2} e^t \cancel{\sin t \cos^2 t} - \left(\frac{1}{2} e^t \cancel{\sin t \cos^2 t} - \frac{1}{4} e^t \sin t\right)$

$= -\frac{1}{2} t e^t \cos t + \frac{1}{4} e^t \sin t$

$y = c_1 e^t \cos t + c_2 e^t \sin t - \frac{1}{2} t e^t \cos t + \frac{1}{4} e^t \sin t$

$= c_1 e^t \cos t + c_2' e^t \sin t - \frac{1}{2} t e^t \cos t$

⑭ $x' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} x + \begin{bmatrix} e^t \\ -4e^t \end{bmatrix}$

$\left.\begin{aligned} x_h &= c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ X(t) &= \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix} \end{aligned}\right\}$ (from example 5)

$X^{-1}(t) = \begin{bmatrix} e^{-3t}/2 & e^{-3t}/4 \\ e^t/2 & -e^t/4 \end{bmatrix}$

$v' = X^{-1} f = \begin{bmatrix} -e^{-2t}/2 \\ 3e^{2t}/2 \end{bmatrix} \Rightarrow v = \begin{bmatrix} -e^{-2t}/4 \\ 3e^{2t}/4 \end{bmatrix}$

$$X_p = Xv = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix} \begin{bmatrix} e^{-2t}/4 \\ 3e^{2t}/4 \end{bmatrix} = \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}$$

$$X = X_h + X_p = \begin{bmatrix} c_1 e^{3t} + c_2 e^{-t} + e^{t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} - e^{t} \end{bmatrix}$$