Math 2243, Midterm Exam 3

December 6, 2001

INSTRUCTIONS: Books and notes are not allowed. Calculators are allowed. Problems 1-3 are in "multiple choice" format. For these problems circle the answer you believe to be correct(only one of the answers listed for each problem is correct). Write *complete solutions* to problems 4-6 for full credit. You have 50 minutes to work on the problems.

Name: ______________________ TA Section: ______________________

1) (10 pts) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation with $T((1,0,1)) = (3,2,1)$, $T((0,1,0)) = (2,2,0)$, $T((0,0,1)) = (-1,-1,-1)$. Then $T((3,3,5))$ is equal to

(A) $(9,6,3)$

(B) $(13,10,1)$

(C) $(15,12,3)$

(D) $(-2,-2,1)$

(E) $(9,6,-2)$

(Hint: Try first to write $(3,3,5)$ as a linear combination of vectors whose T-values you know.)

$$(3,3,5) = 3(1,0,1) + 3(0,1,0) + 2(0,0,1)$$

$$T(3,3,5) = 3(3,2,1) + 3(2,2,0) + 2(-1,-1,-1)$$

$$= (13,10,1)$$

2) (10 pts) Let $W(t)$ be the Wronskian of the vector-functions $\mathbf{x}_1(t) = \begin{bmatrix} e^{-2t} \\ \cos 3t \\ -\sin 3t \end{bmatrix}$,

$\mathbf{x}_2(t) = \begin{bmatrix} 0 \\ \sin 3t \\ \cos 3t \end{bmatrix}$, $\mathbf{x}_3(t) = \begin{bmatrix} e^{-2t} \\ -\cos 3t \\ \sin 3t \end{bmatrix}$. Then

(A) $W(0) = 2$ and the vectors are linearly independent.

(B) $W(0) = 0$ and the vectors are linearly dependent.

(C) $W(0) = 0$ and the vectors are linearly independent.

(D) $W(0) = 2$ and the vectors are linearly dependent.

$W(0) = 2$ and the vectors may be either linearly dependent, or linearly independent.

$$W(t) = \begin{vmatrix} e^{-2t} & 0 & e^{-2t} \\ \cos 3t & \sin 3t & -\cos 3t \\ -\sin 3t & \cos 3t & \sin 3t \end{vmatrix}$$

$$W(0) = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = -(1)(-1-1) = 2$$

3) (10 pts) Consider the differential system $\mathbf{x}' = A\mathbf{x} + \mathbf{b}(t)$, with $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$.

Assume you are given a solution $\mathbf{x}_p(t)$. Then the general solution has the form

(A) $c_1 e^{-t}\mathbf{v}_1 + c_2 e^{-t}\mathbf{v}_2$

(B) $c_1 e^{-t}\mathbf{v}_1 + c_2 e^{-t}\mathbf{v}_2 + \mathbf{x}_p$

(C) $c_1 e^{-t}\mathbf{v}_1 + c_2 e^{-t}(\mathbf{v}_2 + t\mathbf{v}_3) + \mathbf{x}_p$

(D) $c_1 e^{t}\mathbf{v}_1 + c_2 e^{t}\mathbf{v}_2 + \mathbf{x}_p$

(E) $c_1 e^{-t}(\mathbf{v}_1 + t\mathbf{v}_2) + c_2 e^{-t}(\mathbf{v}_3 + t\mathbf{v}_4) + \mathbf{x}_p$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ -1 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 + 2\lambda + 1$$

$$= (\lambda+1)^2$$

$$\lambda = -1, -1$$

End of multiple choice problems

4) (30 points) (*a*) Consider the matrix $A = \begin{bmatrix} 3 & -2 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & 3 \end{bmatrix}$. Determine if A is diagonalizable. In case it is, find a matrix B such that $B^{-1}AB$ is a diagonal matrix.

(*b*) If A is the matrix in part (*a*), find the solution to the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

a) $\det(A - I\lambda) = \begin{vmatrix} 3-\lambda & -2 & -2 \\ 1 & -\lambda & -2 \\ 0 & 0 & 3-\lambda \end{vmatrix}$

$= (3-\lambda)(\lambda^2 - 3\lambda + 2)$

$= (3-\lambda)(\lambda - 1)(\lambda - 2)$

$\lambda = 1, 2, 3$

3 eigenvalues $\Rightarrow$ 3 eigenvectors $\Rightarrow$ diagonalizable

$\lambda = 1$ $\begin{bmatrix} 2 & -2 & -2 \\ 1 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 2$ $\begin{bmatrix} 1 & -2 & -2 \\ 1 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ eigenvectors

$\lambda = 3$ $\begin{bmatrix} 0 & -2 & -2 \\ 1 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

b) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $B^{-1}AB = D$. $A = BDB^{-1}$.

Let $\bar{x} = B^{-1}x$. Then $\bar{x}' = B^{-1}x' = B^{-1}Ax = B^{-1}BDB^{-1}x = D\bar{x}$.

$\bar{x}_1 = c_1 e^{t}$, $\bar{x}_2 = c_2 e^{2t}$, $\bar{x}_3 = c_3 e^{3t}$,

$X = B\bar{x} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{t} \\ c_2 e^{2t} \\ c_3 e^{3t} \end{bmatrix} = \begin{bmatrix} c_1 e^{t} + 2c_2 e^{2t} - c_3 e^{3t} \\ c_1 e^{t} + c_2 e^{2t} - c_3 e^{3t} \\ c_3 e^{3t} \end{bmatrix}$

$X(0) = \begin{bmatrix} c_1 + 2c_2 - c_3 \\ c_1 + c_2 - c_3 \\ c_3 \end{bmatrix} \Rightarrow c_3 = 0, c_2 = 0, c_1 = 1 \Rightarrow X = \begin{bmatrix} e^{t} \\ e^{t} \\ 0 \end{bmatrix}$

5) (25 pts) Find the general solution to the first-order linear differential system

$$\begin{aligned} x_1' &= 5x_1 - 5x_2 \\ x_2' &= 2x_1 - x_2 \end{aligned}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow X' = \begin{bmatrix} 5 & -5 \\ 2 & -1 \end{bmatrix} X = AX$$

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -5 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 5$$

$$= (\lambda - 2)^2 + 1$$

$$= (\lambda - 2 + i)(\lambda - 2 - i)$$

$\lambda = 2 \pm i$

$$\begin{bmatrix} 3 \mp i & -5 \\ 2 & -3 \mp i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenvectors are $\begin{bmatrix} 5 \\ 3 \mp i \end{bmatrix}$.

$$P = \begin{bmatrix} 5 & 5 \\ 3-i & 3+i \end{bmatrix}, \quad D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}$$

$$P^{-1} = \frac{1}{10i} \begin{bmatrix} 3+i & -5 \\ -3+i & 5 \end{bmatrix}$$

$P^{-1}AP = D$

$$X' = AX = PDP^{-1}X \Rightarrow P^{-1}X' = DP^{-1}X$$

Let $\bar{X} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = P^{-1}X$. Then $\bar{X}' = D\bar{X}$.

$$\bar{x}_1 = c_1 e^{(2+i)t}, \quad \bar{x}_2 = c_2 e^{(2-i)t}$$

$$X = P\bar{X} = \begin{bmatrix} 5 & 5 \\ 3-i & 3+i \end{bmatrix} \begin{bmatrix} c_1 e^{(2+i)t} \\ c_2 e^{(2-i)t} \end{bmatrix} = \begin{bmatrix} 5(c_1 e^{(2+i)t} + c_2 e^{(2-i)t}) \\ (3-i)c_1 e^{(2+i)t} + (3+i)c_2 e^{(2-i)t} \end{bmatrix}$$

$$X = \begin{bmatrix} 5e^{2t}(A\cos t + B\sin t) \\ 3e^{2t}(A\cos t + B\sin t) + e^{2t}(A\sin t - B\cos t) \end{bmatrix}$$

6) (25 pts) Find two linearly independent solutions for the first-order linear differential system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$. (Note that the matrix is the same as in problem 3, so you may use any of the calculations you did for that problem - however, you need to write them down here.)

From 3, $\lambda = -1, -1$

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{eigenvector}$$

Solve $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\Rightarrow \quad 2v_1 + 4v_2 = 2$

Set $v_2 = 0 \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Independent solutions are $e^{-t}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $te^{-t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.